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FOUNDATIONS

***Becker, Oskar. Einführung in die Logistik, vorzüglich in den Modalkalkül.** Westkulturverlag Anton Hain, Meisenheim am Glan, 1951. 92 pp.

This booklet is an introduction to the classical logical calculus with special emphasis on modal systems. The four chapters treat respectively Boolean algebra, the algebra of propositions, the first-order predicate calculus, and modal propositional algebra. Since the author [Jahrbuch für Philosophie und phänomenologische Forschung 11, 497–548 (1930)] is himself one of the major contributors to the theory of modal systems, it is natural that there should be an emphasis on that subject; here the first three chapters are intended to form a background for the fourth. The whole has more the character of a summary than a treatise: the author attempts to explain the ideas with some attention, at least in the earlier portions, to their historical evolution; but he gives proofs only in very simple cases, and he does not emphasize the formal character of logical deduction. The effect is readable and interesting. The last chapter is essentially like the treatment of Lewis [in C. I. Lewis and C. H. Langford, Symbolic Logic, Century, New York, 1932] modified so as to be based explicitly on the classical algebra; no account is taken of the semantical studies of Carnap, McKinsey and others, of the relations to intuitionistic systems, of topological applications, or of the uses of the Gentzen technique in connection with modal systems.

H. B. Curry (State College, Pa.).

Zubieta R., Gonzalo. Functional calculi of the first order with an identity. Bol. Soc. Mat. Mexicana 8, 15–21 (1951). (Spanish)

This paper contains a careful formulation of the classical first order predicate calculus with identity and functional (or descriptive) operations; the basic propositional connective is the Sheffer stroke function, and all tautologies are taken, à la Quine, as axioms. The author states four theorems regarding the system, viz.: (1) the deduction theorem; (2) Henkin's completeness theorem [J. Symbolic Logic 14, 159–166 (1949); these Rev. 11, 487]; (3) if P is a class of propositions and q takes the value T in every truth table valuation which verifies P, then q is deducible from P; and (4) an analogous theorem when identity is taken into account. The first theorem is proved rather summarily; for the second the reader is referred to Henkin; while the remaining two are derived simply from the others.

H. B. Curry (State College, Pa.).

Rose, Alan. A formalization of the $C-O$ propositional calculus. Proc. Cambridge Philos. Soc. 47, 635–636 (1951).

This is a formalization, using the Łukasiewicz technique, of the classical propositional algebra based on implication (C) and the constant O (denoting a false statement) as primitives. The axioms consist of the single axiom of Łukasiewicz [Proc. Roy. Irish Acad. Sect. A. 52, 25–33 (1948); these Rev. 10, 93] and the axiom COp. Using previously known results the author shows that his formalization is complete.

H. B. Curry (State College, Pa.).

Rose, Alan. Strong completeness of fragments of the propositional calculus. J. Symbolic Logic 16, 204 (1951).

A simple proof of the theorem: If in a fragment of the propositional calculus material implication can be defined in terms of the primitive functions, then any weakly complete (i.e. having as theorems all expressible tautologies) formalization of the fragmentary system which has for rules of procedure the substitution rule and modus ponens is also strongly complete (from any expressible formula which is not a tautology any expressible formula is deducible).

A. Heyting (Amsterdam).

***Patterson, George W. Logical syntax and transformation rules.** Proceedings of a Second Symposium on Large-Scale Digital Calculating Machinery, 1949, pp. 125–133. Harvard University Press, Cambridge, Mass., 1951. $8.00.

The author begins with an elementary exposition of the basic ideas of semiotics in the sense of Carnap. He then proposes a calculus which, if fully developed, would be an applied first order predicate calculus with two sorts of individuals. The first sort of individuals are the expressions (i.e. finite linear series) of an object language consisting of β distinct symbols (the digits); the operations of concatenation and of cyclic succession of the digits are allowed as descriptive functions; the only predicate is identity. The second sort of individuals are the successive instants of time; there are variable one-place predicates over this range, which are interpreted so that $\xi(t)$, for instance, is to mean that a terminal is in the excited state. (For $\beta=2$ the two values of these functions can be identified with the two characters in the language, and the author appears sometimes to think of these functions as descriptive, but in his formulas he treats them as predicates.) He applies this calculus to the analysis of some extremely simple computing circuits. These are examples of a method which he says can be "extended to more complex situations which are at present difficult to investigate except by our sometimes fallible intuition."

H. B. Curry (State College, Pa.).

Alves, Maria Teodora. A theorem of metamathematics. Gaz. Mat., Lisboa 12, no. 49, 6–8 (1951). (Portuguese)

The author comments on the general law of contraposition, i.e., that in theorems of the form

$$H_1 \wedge H_2 \wedge \cdots \wedge H_n . \supset . T$$

we can replace H_i by $\neg T$ and T by $\neg H_i$ and obtain an equivalent theorem. She proves this by well known methods and gives examples from geometry, arithmetic, and algebra.

H. B. Curry (State College, Pa.).

Martin, R. M., and Woodger, J. H. Toward an inscriptional semantics. J. Symbolic Logic **16**, 191–203 (1951).

Um dem "nominalistischen" Ziel einer wissenschaftlichen Sprache zu dienen, die nur von konkreten Objekten, aber nicht von abstrakten Objekten, wie Sachverhalten, Begriffen, u.ä spricht, wird hier der semantische Wahrheitsbegriff in einer Metasprache M definiert, die nur Variablen $a, b, \cdots$ für die Zeichen (inscriptions) einer Objektsprache L und Variablen $x, y, \cdots$ für Objekte enthält. M hat keine Variablen für Zeichengestalten (sign-designs). Zu jedem a enthält M eine Übersetzung von a und eine strukturelle Beschreibung von a. Diese setzt sich zusammen aus Prädikaten von M mit der Bedeutung z. B. "a hat die Gestalt (", und aus semiotischen Prädikaten wie der Verkettung. Für die Namen a von L und die Prädikate $a=\hat{x}b$ (wenn b als einzige freie Variable x enthält) enthält M ferner ein Prädikat D (denotation) mit der Bedeutung "a bedeutet x" von aDx. Namen haben genau ein Objekt als Bedeutung. Für Prädikate wird $\hat{x}bDx$ als äquivalent mit der Übersetzung von b definiert. Eine Begründung dieses neuartigen Gebrauchs von "Bedeutung" wird nicht gegeben. Anschliessend wird für strukturelle Beschreibungen ein Prädikat TS (true shape) rekursiv definiert, so dass $TS(A)$ äquivalent ist mit der Übersetzung von a, wenn A die strukturelle Beschreibung von a ist. Es wird angekündigt, dass zur Definition des Wahrheitsbegriffes TS keine Rekursion erforderlich sei.

P. Lorenzen (Bonn).

Ridder, J. Formalistische Betrachtungen über intuitionistische und verwandte logische Systeme. V. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 94–105 (1951).

The author continues his discussion of logical calculi of the Gentzen type containing constants ν and λ, to be interpreted respectively as a true and a false proposition. [same Proc. **53**, 1375–1389 = Indagationes Math. **12**, 445–459 (1950); these Rev. **12**, 579]. As he has done earlier for two subsystems of the intuitionistic system, he presents a normal form in the classical and intuitionistic systems for derivations of sequences of the form $\nu \rightarrow A$, where A does not contain either λ or ν. He demonstrates the Hauptsatz (the eliminability of the Schnitt schema and a variation of it) for these propositional and predicate calculi. He discusses known applications of these results to consistency and decision procedure, and, in the case of the intuitionistic system, the non-provability of the principle of excluded middle.

D. Nelson (Washington, D. C.).

Ridder, J. Formalistische Betrachtungen über intuitionistische und verwandte logische Systeme. VI. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 169–177 (1951).

The author presents axioms for modal operators for necessity and possibility to be appended to the axioms of the classical logic and eight subsystems of it discussed in earlier work and reviewed above. The subsystems are: the intuitionistic, the minimal, one for the positive connectives, one for only implication and conjunction, and systems dual to these. Calculi of the Gentzen type are constructed for modal systems for both propositional and predicate calculi. Results on independence of the modal operators are also developed.

D. Nelson (Washington, D. C.).

Ridder, J. Formalistische Betrachtungen über intuitionistische und verwandte logische Systeme. VII. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 226–236 (1951).

Continuing the work reviewed above, the author shows, for various of the systems, a mapping of their theorems into theorems containing N, the modal operator for necessity, in a corresponding modal system. The image formulas in the modal system of the form NX may be interpreted to mean "X is provable". For the duals of the familiar systems a similar result is obtained replacing the operator for necessity by that for possibility. This work depends on the idea of Gödel [Ergebnisse eines math. Kolloquiums **4**, 39–40 (1933)]. The author also restates Gödel's results on the mapping of intuitionistic propositional and predicate calculi and a formal system of arithmetic into the corresponding intuitionistic systems [ibid., 34–38 (1933)]. He also notes similar results for systems dual to the intuitionistic ones.

D. Nelson (Washington, D. C.).

Griss, G. F. C. Negationless intuitionistic mathematics. III. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 193–199 (1951).

The author continues his outline of foundations of negationless intuitionistic mathematics [same Proc. **53**, 456–463 = Indagationes Math. **12**, 108–115 (1950); these Rev. **12**, 3]. In the present section the notions of set, spread, species, and the continuum are discussed. The definition of spread differs only slightly from that of Brouwer [Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1. **12**, no. 5 (1918)]. Only those species are allowed which are subspecies of a previously constructed set, spread, or species. Null species are excluded. The definition of "larger than" receives a formulation not dependent upon negation. The real numbers are introduced as sets of nested intervals of rationals, and alternative definitions are discussed. In Brouwer's work, a real number is said to be different from another just in case the assumption of their equality leads to an absurdity; this concept finds no place in the present theory. The continuum is defined as a species of species of real numbers. A proof is presented for Brouwer's theorem [Math. Ann. **96**, 451–488 (1926)] that any function defined at all points of a unit interval of the continuum is uniformly continuous on that interval.

D. Nelson (Washington, D. C.).

Suetuna, Zyoiti. Über die Grundlagen der Mathematik. J. Math. Soc. Japan **3**, 59–68 (1951).

Foundation of mathematics on the basis of the philosophy of Nishida. Though the general trend is intuitionistic, it is accepted that we possess an "intuition brought about by action" of the totality of natural numbers and of the linear continuum, the latter being "contradictorily self-identical" with the set of all real numbers; the contradiction is seen in the fact that the set of real numbers can be divided by a Dedekind cut, whereas the continuum is indivisible. By a similar intuition we are able to decide in which cases the principle of excluded middle may be applied; e.g. for a given exponent n either natural numbers x, y, z with $x^n+y^n=z^n$ exist or do not exist.

A. Heyting.

Lorenzen, Paul. Die Widerspruchsfreiheit der klassischen Analysis. Math. Z. **54**, 1–24 (1951).

The author outlines a method of constructing a theory of real numbers based on his foundation of classical arithmetic [Math. Z. **53**, 162–202 (1950); these Rev. **12**, 469]. A one-

place predicate (relations similarly) is defined inductively by a set of rules of the form $A_1 \rightarrow p(x_1), \cdots, A_n \rightarrow p(x_n)$, where $A_1, \cdots, A_n$ are statements which may contain the logical symbols introduced in the earlier paper and previously defined predicate symbols, but no bound predicate variables. A symbolism for sets, terms and functions over the integers is introduced. Rationals are introduced in a familiar fashion. Rationals and systems (n-tuples) of them are called entities of the 0-level (Schicht). The entities of the 0-level, statements and systems of statements about these are called entities of the 1-level. In this fashion, by repetition, a hierarchy of levels is introduced, each level containing the entities of all the preceding ones as well as systems of statements about these entities. The union of these levels constitutes the zero hyper-level. Sets of each level may not be countable in that level but are always countable in the next higher one. This fact the author uses to justify a principle of choice. Real numbers are introduced by a symbolism for the greatest lower bound of a set of rationals. It is shown that every set of real numbers has a real number as a greatest lower bound; however, in general, this real number will be an entity of the next higher level. This completeness property is the reason for limiting consideration to the real numbers of the finite levels. The author demonstrates a number of classical theorems on limit points and limits of sequences. Intervals receive a particular treatment since they are sets of the zero hyper-level. A function defined over an interval is also a hyper-level entity. Let f be such a function, suppose that f^k is defined as f except that in each reference to real number we limit attention in the definition to real numbers of the k-level. If for each k, f^k is a function, then f is said to be a quasi-function. For such functions the author defines "limit" and "continuity" and proves fundamental theorems. A final section of the paper discusses sets and sequences of quasi-functions. *D. Nelson.*

Kurepa, Đuro. Über die Prinzipien der Induktion. Srpska Akad. Nauka. Zbornik Radova, Knj. 7. Matematički Institut, Knj. 1, 109–118 (1951). (Serbo-Croatian. German summary)

Verfasser befasst sich mit der Frage wann aus $\pi \subseteq S$ auf die Identität der Mengen π und S geschlossen werden kann. Folgende Fälle werden betrachtet: 1) S ist eine beliebige Menge; 2) S ist teilweise geordnet; 3) S ist geordnet und lückenlos; 4) S ist wohlgeordnet (transfinite Induktion); und schliesslich 5) S ist die Menge der natürlichen Zahlen (totale Induktion). *Author's summary.*

***Freytag gen. Löringhoff, Bruno, Baron v. Philosophical Problems of Mathematics.** Philosophical Library, New York, N. Y., 1951. iv+88 pp. $2.75.

This is a translation of "Gedanken zur Philosophie der Mathematik" [Westkulturverlag, Meisenheim am Glan, 1948] by A. von Zeppelin, and consists of three short papers. The first, Philosophy and Mathematics, is an essay on the interrelations between these two subjects. The second, Philosophical Problems of Mathematics, treats briefly the meaning of "existence" of mathematical objects, and the justification for "applying" mathematics to reality. The last, The Human Aspect of Mathematics, indicates the manner in which mathematics and human beings characterize each other. *C. C. Torrance* (Monterey, Calif.).

***Whittaker, Edmund. Eddington's Principle in the Philosophy of Science.** Cambridge, At the University Press, 1951. v+35 pp.

This paper outlines the evolution of the basic idea underlying Eddington's Principle, and presents a defence against criticisms that have been made of this Principle.

C. C. Torrance (Monterey, Calif.).

ALGEBRA

***Sierpiński, Wacław. Zasady algebry wyższej z przypisem Andrzeja Mostowskiego zarys teorii Galois. [Principles of Higher Algebra with an appendix by Andrzej Mostowski, Outline of Galois Theory].** 2d ed. Monografie Matematyczne. Tom XI. Polskie Towarzystwo Matematyczne, Warszawa-Wrocław, 1951. viii+436 pp. For a review of the first edition see these Rev. 8, 498.

Aquaro, Giovanni. Una dimostrazione del teorema di Cramer indipendente dalla teoria dei determinanti. Boll. Un. Mat. Ital. (3) 6, 240–245 (1951).

Parodi, Maurice. Sur l'irréductibilité du polynome caractéristique de matrices carrées régulières à éléments entiers sur le corps des nombres rationnels. Ann. Soc. Sci. Bruxelles. Sér. I. 65, 15–24 (1951).

Using special similarity transformations the author generalizes his criterion [C. R. Acad. Sci. Paris 231, 97–99 (1950); these Rev. 12, 74] for the irreducibility of the characteristic equation of certain matrices in the field of rational numbers. *A. Brauer* (Chapel Hill, N. C.).

Taussky, Olga. Bounds for characteristic roots of matrices. II. J. Research Nat. Bur. Standards 46, 124–125 (1951).

Let $A=(a_{ik})$ be a non-negative square matrix of order n and C_i the circles with centers a_{ii} and radii $\sum_{i \neq k} a_{ik}$ $(i=1, 2, \cdots, n)$. The author proves that the greatest characteristic root λ of A is greater than or equal to the greatest element of the main diagonal. Using this result it is proved that if A cannot be transformed into the form $\begin{pmatrix} P & 0 \\ U & Q \end{pmatrix}$ by the same permutation of rows and columns, then λ cannot be a common point of the n circles C_i unless it is a common boundary point of at least two of these circles. If A is an arbitrary matrix and ω a characteristic root which lies in the interior or on the boundary of only one of the circles C_i, then there cannot exist two linearly independent characteristic vectors belonging to ω. (In line 20 page 125 the words "positive (non-negative)" should be omitted.)

A. Brauer (Chapel Hill, N. C.).

Odqvist, F. K. G. An expansion of frequency determinants with application to the normal frequencies of a spring mounted rigid body (resilient foundation). Quart. Appl. Math. 9, 441–448 (1952).

The secular determinant is expanded in a linear combination of products of terms from the main diagonal. This expansion is used to obtain power series for the eigenvalues of a general vibration problem and a particular application.

P. Franklin (Cambridge, Mass.).

Wilansky, Albert. The row-sums of the inverse matrix. Amer. Math. Monthly 58, 614–615 (1951).

Bjerhammar, Arne. Rectangular reciprocal matrices, with special reference to geodetic calculations. Bull. Géodésique 1951, 188–220 (1951).

Let the matrix $\mathbf{A}$ have s rows and r columns, where $r \neq s$. When $\mathbf{A}$ has the rank $\min(r, s)$, the author introduces a rectangular reciprocal matrix $\mathbf{A}^{-1}$ with r rows and s columns, depending on a matrix $\mathbf{B}$: When $r<s$, choose $\mathbf{B}$ with s rows and $s-r$ columns such that the square matrix $[\mathbf{AB}]$ is non-singular. Then $\mathbf{A}^{-1}$ is defined as the first r rows of the inverse of $[\mathbf{AB}]$. One has $\mathbf{A}^{-1}\mathbf{A}=I$, the identity matrix of order r, while $\mathbf{A}\mathbf{A}^{-1}=\mathbf{A}^0$, of order s and rank r, has a few properties of an identity matrix. One gets the unique solution of a consistent system $\mathbf{Ax}=\mathbf{L}$ ($\mathbf{x}, \mathbf{L}$=columns) in the form $\mathbf{x}=\mathbf{A}^{-1}\mathbf{L}$, independent of $\mathbf{B}$. For $r>s$ the author defines $\mathbf{A}^{-1}$ as $[(\mathbf{A}')^{-1}]'$, and finds that the underdetermined system $\mathbf{Ax}=\mathbf{W}$ ($\mathbf{x}, \mathbf{W}$=columns) has the general solution $\mathbf{x}=\mathbf{A}^{-1}\mathbf{W}+(\mathbf{A}^0-\mathbf{I})\mathbf{M}$, where $\mathbf{M}$ is an arbitrary column. These rectangular reciprocal matrices are used in an exposition of various procedures for adjusting observations by the method of least squares. *G. E. Forsythe.*

Hall, Marshall, and Ryser, H. J. Cyclic incidence matrices. Canadian J. Math. 3, 495–502 (1951).

Using the theory of difference sets and their multipliers [Hall, Duke Math. J. 14, 1079–1090 (1947); these Rev. 9, 370], the authors obtain some necessary conditions for the existence of a cyclic solution of the combinatorial problem of arranging v elements into v sets such that each set contains exactly k distinct elements while each pair of sets has exactly λ elements in common ($0<\lambda<k<v$). *H. S. M. Coxeter* (Toronto, Ont.).

Drazin, M. P. Some generalizations of matrix commutativity. Proc. London Math. Soc. (3) 1, 222–231 (1951).

Let $\mathfrak{C}^{(0)}$ be a set $\{A_1, \cdots, A_m\}$ of matrices of order n over an algebraically closed field of characteristic zero; and define $\mathfrak{C}^{(k)}=\{A_i C^{(k-1)}-C^{(k-1)}A_i\}$, where $i=1, \cdots, m$ and $C^{(k-1)}$ runs through all members of $\mathfrak{C}^{(k-1)}$. If $\mathfrak{C}^{(k)}=0$, the set $\mathfrak{C}^{(0)}$ is said to have the property Q_k. The set $\mathfrak{C}^{(0)}$ has the property N_k if $p(A_1, \cdots, A_m)C^{(k)}$ is nilpotent for every polynomial $p(A_1, \cdots, A_m)$ and every element $C^{(k)}$ of $\mathfrak{C}^{(k)}$. This paper deals with the relationships between properties Q_k and N_l for various k and l, and also considers related questions having to do with characteristic roots and eigenvectors. The author has overlooked the close connection with the classical theory of Lie algebras. *N. H. McCoy.*

Brenner, J. L. Matrices of quaternions. Pacific J. Math. 1, 329–335 (1951).

A quaternion λ is called a characteristic root of a given square matrix if a non-zero vector x exists such that $Ax=x\lambda$. The author proves that every matrix of real quaternions can be transformed into triangular form by a unitary matrix, and that in this form the characteristic roots are just the diagonal elements and all quaternions similar to them (viz., $\rho^{-1}\lambda\rho$). He points out that the same results were obtained independently by H. C. Lee [Proc. Roy. Irish Acad. Sect. A. 52, 253–260 (1949); these Rev. 12, 153]. It is strange that the next theorem after Theorem 2 is Theorem 10. *H. S. M. Coxeter* (Toronto, Ont.).

Newell, M. J. A theorem on the plethysm of S-functions. Quart. J. Math., Oxford Ser. (2) 2, 161–166 (1951).

In the symbolism used for the plethysm of S-functions the coefficient $g_{\lambda\mu\nu}$ is defined by the equation

$$\{\lambda\} \otimes \{\mu\} = \sum g_{\lambda\mu\nu}\{\nu\}.$$

The principal theorem of the paper is that if

$$\{m\} \otimes \{n\} = \sum \{\nu\}$$

where m and n are integers, then

$$\sum g_{(\alpha)(\beta)}\{\xi\} = [\{m-1\} \otimes \{1^k\}][\{m\} \otimes \{n-k\}].$$

In an unfortunate misprint the left hand side is incorrectly written as $\sum g_{(\alpha)(\beta)}\{\nu\}$. A second theorem gives the conjugate result. If

$$\{m\} \otimes \{1^n\} = \sum \{\nu\},$$

then for any integer $k \leqq n$

$$\sum g_{(\alpha)(\beta)}\{\xi\} = [\{m-1\} \otimes \{k\}][\{m\} \otimes \{1^{n-k}\}].$$

These two theorems are used to give an unambiguous method of obtaining the expansions of $\{m\} \otimes \{n\}$ and $\{m\} \otimes \{1^n\}$. The partitions involved in the expansions have $\leqq n$ parts. Those partitions which have exactly n parts are exactly determined by the first theorem taking $k=n$. The case $k=n-1$ then determines exactly the partitions which have exactly $(n-1)$ parts, and so on. *D. E. Littlewood.*

Abstract Algebra

Chin, Louise H., and Tarski, Alfred. Distributive and modular laws in the arithmetic of relation algebras. Univ. California Publ. Math. (N.S.) 1, 341–384 (1951).

A relation algebra is a boolean algebra with additional operations of "relative multiplication" $(a|b)$ and "conversion" (a^{-1}), characterized by a set of axioms [cf. Tarski, J. Symbolic Logic 6, 73–89 (1941); these Rev. 3, 130] all of which are valid in every algebra of binary relations. In part one, consequences of these axioms are derived, with special attention to equivalence elements (transitive and symmetric), functional elements ($a^{-1}|a \subset I$, the "identity"), and ideal elements ($V|a|V=a$, for V the "universal element"). In part two the left (and corresponding right) distributive laws $a\cdot(b+c)=a\cdot b+a\cdot c$, strong modular laws $a\cdot(b+c)=a\cdot b+c$, and standard modular laws $a\cdot(b+a\cdot c)=a\cdot b+a\cdot c$, are investigated, for all choices of $\cdot$ and $+$ from the operations union, intersection, multiplication, and the boolean dual of multiplication. In particular, where a law does not hold or fails identically in every non-trivial algebra, conditions on a are found for the law to hold for all b and c. These conditions are roughly of the type that a be I or V, or that a be an equivalence, functional, or ideal element. *R. C. Lyndon.*

Evans, Trevor. On multiplicative systems defined by generators and relations. I. Normal form theorems. Proc. Cambridge Philos. Soc. 47, 637–649 (1951).

This paper is concerned with properties of multiplicative systems in which the associative law is not assumed. While most of the discussion is devoted to loop theory, the author points out that many of the results go through, with minor changes, for groupoids, quasigroups, etc. The author uses the definition of a loop in terms of three binary operations introduced by him in a previous paper [J. London Math. Soc. 24, 254–260 (1949); these Rev. 11, 327]. The concept of a set of relations in closed form is presented and normal-form theorems are deduced for loops defined by generators and a closed set of relations—thus, in particular, for free loops and free products of loops with or without amalgamated subloops. These general results are then applied to yield immediately an embeddability theorem for loops and to give a complete characterization of the structure of sub-

loops of a loop generated by a closed set of relations. Finally, the author uses the normal form theorems, together with a special case of results obtained by him in a preceding paper [ibid. 26, 64–71 (1951); these Rev. 12, 475] to give solutions to the word problem for loops and to the cancellation and division problems for groupoids. *G. Bates.*

Iseki, Kiyoshi. Structure of special ordered loops. Portugaliae Math. 10, 81–83 (1951).

A loop A with operation $+$ is said to be an ordered loop if it is simply ordered under a relation $>$ with the additional property: $a>b$ implies $c+a>c+b$ and $a+c>b+c$ for every c in A. The nth right addition nx of x is defined inductively by $1\cdot x=x$ and $nx=(n-1)x+x$. An element $a>0$ is said to be Archimedean (right Archimedean) if for every b in A there exists a positive n such that $na\geqq b$. If all positive elements of A are Archimedean we say that A is Archimedian. An atom is an element $a>0$ such that $a>b\geqq 0$ requires $b=0$. If an ordered loop A has an atom which is Archimedean then it is a cyclic group. An ordered loop A is called conditionally σ-complete if every nonvoid countable bounded set has a greatest lower bound and a least upper bound. Every conditionally σ-complete ordered loop is Archimedean. An ordered loop is a normal Hausdorff space under its interval topology and the functions $a+b$, $a-b$, and $-a+b$ are continuous in a and b. An ordered loop is connected in the interval topology if and only if it is Archimedean. *R. M. Thrall* (Ann Arbor, Mich.).

Maeda, Fumitomo. Direct sums and normal ideals of lattices. J. Sci. Hiroshima Univ. Ser. A. 14, 85–92 (1950).

In a direct decomposition of a lattice with zero element the component lattices are neutral ideals of the original lattice. The writer uses this to dispense with the assumption of a unit element in the von Neumann-Iwamura discussion of a reducible continuous geometry. Let L be a lattice with zero element such that $L(0, a)$ is a continuous geometry, possibly reducible, for every a in L. For every set S of elements of L let S^∇ be the set of elements which are distributive with each element of S. Then $S^{\nabla\nabla\nabla}=S^\nabla$ and the S^∇ are precisely all neutral ideals of L. Those which contain maximal elements are precisely the $L(0, z)$ with z in the centre of L. These neutral ideals form a complete Boolean algebra Z and L is irreducible if and only if Z consists of (0) and L. Let the maximal ideals P of Z be considered as the points of a space Ω. Then L is isomorphic to part of a direct sum $\sum_\Omega L(P)$ where each $L(P)$ is an irreducible continuous geometry, possibly without a unit element. To each a in L corresponds a dimension function $D_a(P)$ such that for fixed P the values of $D_a(P)$ range over all integers less than k with $k\leqq\infty$, or over all real numbers less than or equal to a finite positive c, or over all real non-negative numbers. *I. Halperin* (Kingston, Ont.).

Neumann, B. H. Embedding non-associative rings in division rings. Proc. London Math. Soc. (3) 1, 241–256 (1951).

Let $\mathfrak{R}$ be a non-associative ring. If the equations $ax=b$ and $ya=b$ have solutions for x and y whenever $a\neq 0$, then $\mathfrak{R}$ is a division ring. If, furthermore, these solutions are unique (or, equivalently, if $\mathfrak{R}$ has no divisors of zero), then $\mathfrak{R}$ is a unique-division ring. The ring $\mathfrak{R}$ can be imbedded in a division ring $\mathfrak{R}^*$ if and only if all non-zero elements of $\mathfrak{R}$ have the same characteristic, and this characteristic is either zero or a prime. The ring $\mathfrak{R}$ can be imbedded in a unique-division ring $\mathfrak{R}^*$ if and only if $\mathfrak{R}$ has no divisors of zero. If $\mathfrak{R}$ is commutative (or idempotent), then $\mathfrak{R}^*$ may, in each theorem, be taken to be commutative (or idempotent). Conditions that $\mathfrak{R}^*$ have a unit element are given. *F. Kiokemeister* (Chicago, Ill.).

Nagata, Masayoshi. Some studies on semi-local rings. Nagoya Math. J. 3, 23–30 (1951).

A semi-local ring is a commutative ring with identity containing only a finite number of maximal ideals $M_1, \cdots, M_r$ and such that $\bigcap_{n=1}^{\infty}(M_1\cdots M_r)^n=(0)$. Section 1 concerns a semi-local ring R which is a subdirect sum of semi-local rings and considers conditions under which R is a subspace of the direct sum. In section 2 it is proved that a Noetherian semi-local ring is complete if R/p is complete for every minimal prime divisor p of (0). Section 3 considers rings containing a given semi-local ring R and contained in the completion of R. *I. S. Cohen* (Cambridge, Mass.).

Asano, Keizo. Über kommutative Ringe, in denen jedes Ideal als Produkt von Primidealen darstellbar ist. J. Math. Soc. Japan 3, 82–90 (1951).

Let a commutative ring R with identity element be called a Dedekind ring if it is the direct sum of a finite number of Dedekind integral domains and of rings having a nilpotent, principal, maximal ideal. Various conditions on R are proved equivalent to its being Dedekind, among them the following: (1) Every ideal in R is a product of prime ideals; (2) the zero ideal is a product of prime ideals, and if a prime ideal P contains an ideal A, then P is a factor of A. In the presence of the ascending chain condition, the following are also equivalent: (3) For every maximal ideal M, there is no ideal between M and M^2; (4) the lattice of ideals is distributive. These results generalize known conditions for an integral domain to be Dedekind. [Rings satisfying (1) have been studied by S. Mori, J. Sci. Hirosima Univ. Ser. A. 10, 117–136 (1940); these Rev. 2, 121.] *I. S. Cohen.*

Nakayama, Tadasi. A remark on finitely generated modules. Nagoya Math. J. 3, 139–140 (1951).

The author proves the following theorem. Let R be a ring, M a finitely generated R-module, such that for any set of generators $u_1, \cdots, u_n$ and any regular maximal right ideal I, $u_1I+\cdots+u_nI=M$. Then $M=0$. This is a generalization of a result of Azumaya [Nagoya Math. J. 2, 119–150 (1951); these Rev. 12, 669]. *I. Kaplansky.*

Mills, W. H. A theorem on the representation theory of Jordan algebras. Pacific J. Math. 1, 255–264 (1951).

If Φ is a field of characteristic p, and m and n are any two integers in Φ, there is an attached integer $s(m, n)$ depending only on p such that $s(m, n)=m+n-1$ if $p=0$ or $p\geqq m+n-1$. Let J be a Jordan algebra over Φ. Then a generalized representation of J is a linear mapping $a\to S_a$ of J into an associative algebra U such that

$$S_aS_bS_c=S_cS_bS_a+S_{(ab)c}=S_aS_{bc}+S_bS_{ac}+S_cS_{ab}.$$

Consider a polynomial $f(\lambda)$ with coefficients in Φ and zero constant term and write $f(\lambda)=\prod_i(\lambda-\alpha_i)^{n_i}$ where α_i are the distinct elements in a splitting field of $f(\lambda)$. Write $\psi(\lambda)=\text{l.c.m.}_{i,j}f(\lambda-\alpha_i/2-\alpha_j/2)^{s(n_i,n_j)}$. The author shows that if $f(a)=0$ then $\psi(S_a)=0$. Moreover, if the algebra generated by the S_a is the universal associative algebra of J, if a generates J, and if $f(\lambda)$ is the minimum function of a, then $\psi(\lambda)$ is the minimum function of S_a. *A. A. Albert.*

Knobloch, Hans-Wilhelm. Verschränkte Produkte aus galoisschen Algebren und ihren Gruppen. Math. Nachr. 6, 11–20 (1951).

Suppose that K/Ω is a Galois algebra in the sense of Hasse, that is, a finite-dimensional, commutative and semi-simple ring with a finite group $\mathfrak{G}$ of automorphisms over Ω so that K considered as a Galois module with respect to $\mathfrak{G}$ is operator-isomorphic to the group ring G of $\mathfrak{G}$ over Ω. The author develops ab ovo, using the now customary module-theoretic approach, the theory of central simple algebras A/Ω which have K for a maximally commutative subring, and he is thus lead to a reformulation of the theory of crossed products. Although most of the results are not new, the author's presentation brings about a simplification of the proofs of theorems like the product formula for crossed product algebras for fields K/Ω; this is largely achieved by the advantageous fact that similarity of algebras is replaced by isomorphism as the consequence of a single reduction theorem for crossed product algebras with respect to the Galois algebra K/Ω to crossed products split by a simple component K_0/Ω of K. In this manner the often repetitious construction of matrix units is reduced considerably. As historical facts the following may be noted here: (i) A quite systematic use, though not under the present name, of Galois algebras occurred first in Hasse's work on the localization of algebras over algebraic number fields (passage to the p-adic completion of the base field with the subsequent potential reduction of the given normal extension to a semi-simple algebra); and (ii) the cyclic case was given prominence by A. A. Albert, with the same idea of simplification, in his Colloquium Lectures of the American Mathematical Society for the year 1939 [Structure of algebras, Amer. Math. Soc. Colloq. Publ., v. 24, New York, 1939; these Rev. 1, 99]. *O. F. G. Schilling.*

Knobloch, Hans-Wilhelm. Zur Kennzeichnung galoisscher Algebren mit vorgegebener Galoisgruppe. Math. Nachr. 6, 21–44 (1951).

Hasse showed in his extension of the Kummer theory and generalization of the Lagrange resolvent that the Galois algebras K/Ω (for some of the notation see the preceding review) can be characterized by factor sets C of the characters and representations of the automorphism group $\mathfrak{G}$ in Ω; for details see H. Hasse [J. Reine Angew. Math. **187**, 14–43 (1949); these Rev. **11**, 576]. In Hasse's treatment the detailed conditions for the equivalence of the prerequisite factor sets C are formally somewhat bothersome; in this paper a new approach is presented which employs the product ring $A\times G$ of the full matrix algebra $A=\Omega_g$, $g=[\mathfrak{G}:1]$, and the group ring G in an essential manner; the ring $K\subset A$ is now considered as a fixed representation of a given Galois algebra $\mathfrak{K}/\Omega$, and the automorphisms $S\in\mathfrak{G}$ of K are described by inner transformations with regular elements $u_S\in A$. Then $\mathfrak{w}=\mathfrak{w}(x)=\sum_S(u_S^{-1}xu_S)S^{-1}$, $S\in\mathfrak{G}$, with $u_S^{-1}\mathfrak{w}u_S=S\mathfrak{w}$ is called the resolvent of $x\in K$. The author then develops the involved formal properties of these resolvents and shows in detail how the resolvents transform for various Γ-representations K of the fixed algebra (equivalence with respect to those automorphisms of A which leave fixed the elements of $\Gamma=\sum\Omega u_S$). The complicated transformation rules are subsequently simplified by using three different representations of the commutator ring $\tilde{\Gamma}$ of Γ in A (an anti-isomorphic representation of G) in the group ring $G\times\bar{G}$ of the cartesian product of $\mathfrak{G}$ by itself. This leads to the characterization of a Galois algebra (§8, p. 32) by a suitably defined class of regular elements in $G\times\bar{G}$; in this construction the resolvent $\mathfrak{w}$ is mapped on an element of $G\times\bar{G}$. Then the analogue of Hasse's commutativity and associativity laws (and the condition for semi-simplicity) are rephrased (factor bases of C in $\mathfrak{K}$ intervene, see Hasse, loc. cit.) by systems of equations (and an inequality) in $G\times\bar{G}$. Finally the author presents a significant application of his algebraic theory to algebraic number fields and obtains an extremely noteworthy generalization of Kummer's explicit decomposition law (determination of the local behavior of the extension by means of the generating radical) for prime divisors $\mathfrak{p}$ which do not divide g and which are unramified. Thus let θ denote the generator of the normal basis of K/Ω which corresponds in the above theory to the invariant class of K, suppose that a is the factor (in G) in $\mathfrak{w}(\theta^x)=\mathfrak{w}(\theta)a$, where x is the absolute norm of $\mathfrak{p}$, sp denotes the linear extension of the reduced trace of A/Ω to $A\times G/G$ by coefficient extension with G, $(K/\mathfrak{p})$ is the Artin class of $g_\mathfrak{p}$ conjugate Frobenius automorphisms. Then

$$(g/g_\mathfrak{p})\sum_{F\in(K/\mathfrak{p})}F\equiv \mathrm{sp}\,[\mathfrak{w}(\theta)\cdot a\cdot\mathfrak{w}(\theta)^{-1}]$$
$$=\mathrm{sp}\,[\mathfrak{w}(\theta^x)\mathfrak{w}(\theta)^{-1}]\pmod{\mathfrak{p}}.$$

O. F. G. Schilling (Chicago, Ill.).

Eilenberg, Samuel, and MacLane, Saunders. Homology theories for multiplicative systems. Trans. Amer. Math. Soc. 71, 294–330 (1951).

A multiplicative system, with a unit element 1 is a set M in which there is defined a binary operation $M\times M\to M$, written $(m_1, m_2)\to m_1m_2$, such that $1m=m=m1$. A free multiplicative system F is a multiplicative system, together with a fixed sequence of elements (generators) $g_1, g_2, \cdots, g_i\neq 1$, such that, if $a_1, a_2, \cdots$ is any sequence of elements of F, there is one and only one endomorphism α of F for which each $\alpha(g_i)=a_i$. A multiplicative system M is said to belong to the free system F if, for every sequence $m_1, m_2, \cdots$ in M, there is one and only one homomorphism of F into M which sends each g_i into m_i. With a free system F there is associated the category $\mathfrak{M}(F)$ consisting of all multiplicative systems $M, N, \cdots$ belonging to F, and all homomorphisms $M\to N$ of one such system into another. There is a universal free system, F_I, whose elements are the (parenthesized) "words" in the generators g_i, and where the multiplication is juxtaposition. $\mathfrak{M}(F_I)$ is then the category of all multiplicative systems, and all free systems that are defined by identities can be obtained by introducing suitable congruence relations in F_I. The free systems are subjected to the following finiteness condition: Let θ_n be the endomorphism of F for which $\theta_n(g_i)=g_i$, for $i\leqq n$, and $\theta_n(g_j)=1$, for $j>n$. The condition is that, for each $x\in F$, there should exist an n for which $\theta_n(x)=x$. A generic word in F_I is one which contains each generator g_i at most once. An element of an arbitrary free system is called generic if it is the image of a generic word in F_I by the canonical homomorphism $F_I\to F$. An endomorphism of F is called generic if it carries disjoint generic elements into disjoint generic elements. A set Φ of endomorphisms of F is called admissible if it is closed under composition and contains every generic endomorphism. A finite sequence $x_1, x_2, \cdots, x_n$ in F is called a Φ-sequence if the endomorphism which carries each g_i into x_i $(i=1, 2, \cdots, n)$, and the remaining g_j into 1, belongs to Φ.

If F is a free system and Φ an admissible set of endomorphisms of F, a Φ-construction K on $\mathfrak{M}(F)$ is a function which assigns to each $M\in\mathfrak{M}(F)$ an abstract cell complex $K(M)$, with chain groups $C_q(K(M))$, $q\geqq 1$, whose generating

q-cells are divided into types t and, for each t, are $r(t)$-tuples of elements of M. It is further required that, if α is a homomorphism of M_1 into M_2, the induced homomorphism $K(\alpha)$ of $K(M_1)$ into $K(M_2)$ should commute with the boundary operator ∂. Finally, it is required that the cells in which the entries form Φ-sequences should generate a subcomplex $K(F, \Phi)$ of $K(F)$.

Let $C(M)$ denote the free abelian group generated by elements $\{x\}$, one for each $x \in M$, and let $B(M)$ denote the subgroup generated by the elements $\{y\}-\{yx\}+\{x\}$. Put $H(M)=C(M)/B(M)$. An augmentation $\bar{K}$ of K is obtained by adjoining a set of homomorphisms $\partial'_M: C_1(K(M)) \to H(M)$, such that $\partial'_M \partial$ maps $C_2(K(M))$ into (0), ∂'_F carries $C_1(K(F, \Phi))$ into the subgroup $H(F, \Phi)$ of $H(F)$ which is generated by the cosets $\{x\}+B(M)$ in which x is a Φ-sequence, and which satisfy the condition $\partial'_{M_2} K(\alpha) = H(\alpha)\partial'_{M_1}$, for every homomorphism $\alpha: M_1 \to M_2$, with $H(\alpha)$ the induced homomorphism $H(M_1) \to H(M_2)$.

A map f of K_1 into K_2 is a family of chain maps $f_M: K_1(M) \to K_2(M)$ satisfying (in addition to the commutativity with the boundary) the natural commutativity relations with respect to the $K(\alpha)$'s, and the relations $\partial'_M = \partial'_M f_M$. The chain map f_F is required to carry $K_1(F, \Phi)$ into $K_2(F, \Phi)$. A homotopy D between two such maps f^0 and f^1 is defined similarly as a family of homotopies D_M between f^0_M and f^1_M.

An augmented Φ-construction $\bar{K}$ is called Φ-acyclic if the homology groups $H_q(K(F, \Phi))$, for $q=0, 1, \cdots$, are all (0). The first main result is the following comparison theorem: Let $\bar{K}$ and $\bar{L}$ be two augmented Φ-constructions on the category $\mathfrak{M}(F)$, and suppose that $\bar{L}$ is Φ-acyclic. Then there is a map f of $\bar{K}$ into $\bar{L}$, and any two such maps are homotopic. The second principal result is the following existence theorem: For any free multiplicative system F and any admissible set Φ of endomorphisms of F, there exists a Φ-acyclic augmented construction $\bar{K}$ on the category $\mathfrak{M}(F)$.

It follows from the first theorem that the homology and cohomology groups of $\bar{K}(M)$ are—up to isomorphisms—independent of the particular choice of the construction $\bar{K}$. This provides a powerful tool for homology theory.

For a more restricted class of free systems, including those defined by the associativity relation, or by the commutative relation, or by both, the above results are extended to constructions with normalization subcomplexes $K_N(M)$ which are generated by cells whose entries are 1 at certain prescribed places. Acyclicity is then demanded only modulo the normalization subcomplex and the results are proved for the K/K_N. It is shown that the homology theory derived from K/K_N amounts to the same as that derived from K.

There are two important choices for the set Φ of endomorphisms: Φ=the set of all endomorphisms; or Φ=the set of all generic endomorphisms. Corresponding constructions are given in detail and additional results are proved for the following categories: 1) all multiplicative systems; 2) all associative systems; 3) all commutative systems; 4) all systems that are both commutative and associative; 5) all groups; 6) all commutative groups. In particular, additivity theorems: $H_q(M+M')=H_q(M)+H_q(M')$ and similarly for cohomology, are obtained for 4) and 6).

Finally, indications are given concerning the modifications that have to be made if the existence of a unit element is not assumed.

G. Hochschild (New Haven, Conn.).

Jacy Monteiro, L. H. Derivations of a field. Bol. Soc. Mat. São Paulo 2, no. 2 (1947), 7–36 (1949). (Portuguese)

This is mainly an expository article on derivations of a field, most of the results being well-known theorems of A. Weil, O. Zariski and S. MacLane. The most original part of the paper concerns the case of separable extensions in the sense of N. Bourbaki (extensions "preserving p-independence", introduced by S. MacLane); the author proves that they are exactly the extensions in which any derivation of the subfield can be extended; from that criterion he deduces new proofs of results of S. MacLane on the existence of separating transcendence bases. *J. Dieudonné.*

✓ ***Schilling, O. F. G. The Theory of Valuations.** Mathematical Surveys, No. 4. American Mathematical Society, New York, N. Y., 1950. vii+253 pp. $6.00.

Since the appearance in 1913 of Kürschak's paper giving for the first time an axiomatic definition of a valuation, a great deal of work has been done on this concept. The present book is the first systematic account of the general theory of valuations and of its applications to certain selected branches of algebra. Due, undoubtedly, to limitations of space, certain topics are omitted entirely. Valuations of algebraic function fields, for example, are not treated. Archimedean valuations are not even mentioned, and hence the global theory of algebraic number fields is also not treated.

We now present a summary by chapters. In Chapter 1, the basic definition is given: A valuation of a division ring is a homomorphism of its multiplicative group into a linearly ordered (not necessarily Abelian) group such that the strong triangle inequality is satisfied. General properties are developed, including elementary facts on extension of valuations, on the ramification index, and the residue class degree. Chapter 2 is on complete fields. First the familiar rank 1 case is considered and Hensel's lemma proved; then follows for arbitrary rank the ideas of completeness, maximal completeness, and pseudoconvergence. It is then shown that a valuation of a complete field F can be extended to fields (or even certain division algebras) which are algebraic over F. Chapter 3 is on the ramification theory. As in the theory of algebraic number fields, there is defined a series of subfields between a given relatively complete field F and an algebraic extension of F, the extension for the most part being assumed normal (but not of finite degree) over F. In the normal case, these intermediate fields can be defined in terms of certain subgroups of the Galois group, and their relationship to the value groups and residue fields is studied.

The next three chapters are of a more special nature. Chapter 4, entitled Special Ideal Theory, treats valuation theoretically first the theory of quasi-equality, then the theory of integral domains with Dedekind ideal theory. Valuations of normal extensions of a field F are now considered. In contradistinction to the situation in the previous chapter, F is not assumed relatively complete. The valuation need no longer extend uniquely and its decomposition group is now studied. In Chapter 5, Arithmetic of Simple Algebras, the main object is the extension of the Dedekind-Noether ideal theory from a given field F to simple algebras A over F. This involves a preliminary discussion of the scalar extension $A \times F_*$, where F_* is the completion of F with respect to one of the valuations V defining the ideal theory in F. Chapter 6, Local Class Field Theory, deals with a field F which is complete with respect to a discrete, rank 1, valuation and

whose residue field is perfect and admits for each n exactly one cyclic extension of degree n. The treatment follows that of Chevalley for the p-adic case and accordingly leans heavily on the theory of algebras. The chapter ends with generalizations of the theory in various directions.

In Chapter 7, after characterizing those topological division rings which are separable, locally compact, and totally disconnected, the author returns to general valuations and takes up the problem of the structure of complete fields.

Each chapter is followed by a bibliography. As has been indicated, the Galois theory of infinite extensions and the theory of linear associative algebras has been extensively used. There are two appendices giving the needed results in these theories. The book ends with a glossary of terms from group, ring, and field theory. There is unfortunately no index.

The book may turn out to be somewhat difficult reading for students or for those who merely want to get an idea as to the methods and main results of the various theories presented. Theorems and lemmas tend to be stated with the utmost possible generality. It is certainly of value to the expert to know the extent to which hypotheses can be weakened, but this generality will perhaps impede the progress of the general reader. Often the conditions on the particular fields, groups, etc., which are considered are stated in passing but are not given in the formal statement of the theorems, with the result that some searching through preceding paragraphs is necessary to determine just what these theorems say.

From the summary given above it can be seen that the book covers a wealth of material heretofore to be found only in scattered papers. To the specialist, it will prove invaluable not merely because of this extensive coverage, but also because of the many references to the literature, where the reader may find alternative treatments or extensions of material of the text. *I. S. Cohen.*

Theory of Groups

Frame, J. S. Characteristic vectors for a product of n reflections. Duke Math. J. **18**, 783–785 (1951).

The group $[3^{n,p,q}]$ is generated by reflections

$$R_1, R_2, \cdots, R_{n+p+q+1}$$

in hyperplanes, every two of which are at right angles except the following pairs: the ith and $(i+1)$th ($i=1, 2, \cdots, n+p$; $n+p+2$, $n+p+3, \cdots, n+p+q$) and the $(n+1)$th and $(n+p+2)$th. These exceptional pairs are inclined at $\pi/3$, so that the reflections in them satisfy relations such as

$$R_1R_2R_1=R_2R_1R_2,$$

whereas all the other pairs commute, e.g., $R_1R_3=R_3R_1$. [See Coxeter, Proc. Cambridge Philos. Soc. **30**, 466–482 (1934).] This is a finite group in Euclidean $(n+p+q+1)$-space if $npq<n+p+q+2$, an infinite group in Euclidean or hyperbolic $(n+p+q)$-space if $npq\geqq n+p+q+2$. The author proves that the product

$$R_1R_2\cdots R_{n+p+q+1}$$

has the characteristic equation $\phi_{n+p+q+1}-\lambda^2\phi_{n-1}\phi_{p-1}\phi_{q-1}=0$, where

$$\phi_r=1+\lambda+\lambda^2+\cdots+\lambda^r.$$

Using a coordinate system in which the rth coordinate is the distance from the rth hyperplane, he finds, for each characteristic root λ, the characteristic vector

$$(-\phi_n^{-1}, -\lambda\phi_n^{-1}, \cdots, -\lambda^n\phi_n^{-1}, \phi_p^{-1}, \lambda\phi_p^{-1}, \cdots, \lambda^{p-1}\phi_p^{-1}, \phi_q^{-1}, \lambda\phi_q^{-1}, \cdots, \lambda^{q-1}\phi_q^{-1}).$$

H. S. M. Coxeter (Toronto, Ont.).

Segal, I. E. The non-existence of a relation which is valid for all finite groups. Bol. Soc. Mat. São Paulo **2**, no. 2 (1947), 3–5 (1949).

The author gives a new proof of the following theorem which was obtained previously by von Neumann and Wigner and by the reviewer: In a free group, the intersection of all normal subgroups with finite indices is the identity. For the proof, the group of all permutations of an arbitrary (infinite) set is suitably topologized and the continuity of this topological group is essentially used. We note that there is also an elementary proof of the same theorem given by T. Tannaka (published in Japanese). *K. Iwasawa.*

Neumann, B. H. Groups with finite classes of conjugate elements. Proc. London Math. Soc. (3) **1**, 178–187 (1951).

A group is said to be an *FC*-group if every class of conjugates in G is finite. In an *FC*-group G the elements of finite order form a characteristic subgroup P and the factor group G/P is abelian and has no element of finite order (except the unity element). If G is finitely generated, P is finite. If every generator of G is of finite order then $G=P$. If G is an arbitrary group then an element g in G is contained in a finite normal subgroup of G if and only if (i) g has finite order and (ii) the class of conjugates of g in G is finite. If for any integer n the set of elements of order n in a group G is finite then the elements of order n generate a finite characteristic subgroup of G. Every element of a group G belongs to a finite normal subgroup if and only if G is an *FC*-group in which every element has finite order. If the index of the center Z of a group G is finite then the commutator subgroup G' is finite. If the commutator subgroup of a group G is finite then G is an *FC*-group. If a group G has k generators and if the commutator subgroup G' has finite order h then the index of the center Z is bounded by h^k.

R. M. Thrall (Ann Arbor, Mich.).

Szele, Tibor. Gruppentheoretische Beziehungen bei gewissen Ringkonstruktionen. Math. Z. **54**, 168–180 (1951).

An Abelian group is called a nil-group if it occurs as additive group of only one ring: the trivial "zero-ring" in which all products vanish. The author has shown previously [Math. Ann. **121**, 242–246 (1949); these Rev. **11**, 496] that mixed nil-groups do not exist, and that torsion-groups are nil-groups if and only if they are direct sums of groups of type (p^∞), for various prime numbers p. (The determination of all torsion-free nil-groups is an open problem.) In the present paper the author determines all Abelian groups which occur as additive groups of exactly one non-trivial ring. The following types are exhaustive: (i) the additive group of rationals, (ii) the cyclic groups of order p, p a prime number, (iii) the direct sum of a cyclic group of order p and any number of groups of type (q^∞) for arbitrary prime numbers $q\neq p$, (iv) the direct sum of a cyclic group of order p and any torsion-free nil-group B with $pB=B$. The author assigns to an Abelian group G the "Nil-Stufe" k, if G occurs as additive group of some ring in which there exist non-vanishing products of k factors, while products of $k+1$

factors vanish in all rings with G as additive group. He proves then that a torsion-group cannot have a Nil-Stufe $\neq 1$, whereas torsion-free groups can have any Nil-Stufe $\geqq 1$, and mixed groups any Nil-Stufe $\geqq 2$.

K. A. Hirsch (London).

Ito, Noboru. Note on (*LM*)-groups of finite orders. Kōdai Math. Sem. Rep. **1951**, 1–6 (1951).

The author proves a number of theorems on finite groups whose subgroup lattice is lower semi-modular. We mention only a few of them. (1) Call a group p-nilpotent when it has a normal p-Sylow-complement. Then G is p-nilpotent, if every proper subgroup is p-nilpotent and the order of G contains apart from p two or more prime factors. If it contains only one, q, then the only type for which the assertion is not true is the product of a normal p-Sylow-subgroup and a cyclic non-normal group of order q. (2) Call a group a C-group if every maximal subgroup of any subgroup has prime index. Then a C-group is p-nilpotent for the smallest prime factor of its order, and it is soluble, indeed "supersoluble" i.e. it has a chief series with factors of prime order [see also O. Ore, Duke Math. J. **5**, 431–460 (1939); G. Zappa, ibid. **6**, 511–512 (1940); these Rev. **2**, 1]. The converse is also true. (3) Call a finite group an LM-group, if every intersection of any two distinct maximal subgroups of any subgroup is maximal in each of these maximal subgroups. Such groups are C-groups, but not conversely. If the order of a C-group G is divisible by at least four primes and if every proper subgroup of G is an LM-group, then so is G. If G is a soluble group, then it is a C- (or an LM-) group, if and only if the factor-group of the hyper-centre is a C- (or an LM-) group. The remaining theorems refer to rather special circumstances. *K. A. Hirsch* (London).

***Baer, Reinhold. The cohomology theory of a pair of groups.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 15–20. Amer. Math. Soc., Providence, R. I., 1952.

The cohomology theory is examined of a "product" $Q=RS=SR$ of disjoint subgroups, with commutation rule $sr^s=rs^r$ for suitable operators [see Zappa, Atti II Congresso Un. Mat. Ital., Bologna, 1940, pp. 119–125 (1942); these Rev. **8**, 367]. Condition (III.R) is that there be a homomorphism of S into R preserving operation on the coefficient module (henceforth tacitly fixed). This holds, for example, if S is a (replica of a) normal subgroup of R, with $r^s=r$ and $s^r=r^{-1}sr$. Three results are announced as typical. (A) Under (III.R) and $r^s=r$, if $H^i(S)=0$ for all $0\leqq i\leqq n$, then $H^n(R)=0$; (B) if also, symmetrically, (III.S) and $s^r=s$, then also $H^{n+1}(R)=H^{n+1}(S)$; (C) a modification of (A) without $H^0(S)=0$. The indicated method of proof is by what J. H. C. Whitehead [Quart. J. Math., Oxford Ser. (2) **1**, 219–228 (1950); these Rev. **12**, 156] calls vector cocycles.

R. C. Lyndon (Princeton, N. J.).

Hönig, Chaim Samuel. Classification of the additive groups of rational numbers. Bol. Soc. Mat. São Paulo **3**, no. 1–2 (1948), 37–47 (1951). (Portuguese)

The characteristic sequence (n) of an additive group G of rationals (other than $G=\{0\}$) is defined by $n_r=\max_{x\in G}(m_r)$, $x=p_1^{-m_1(x)}p_2^{-m_2(x)}\cdots$, and (p) is the sequence of primes. Every term is an integer or $+\infty$, and only a finite number of terms are strictly negative. Each sequence satisfying these conditions is the characteristic sequence of one and only one additive group of rationals. We identify all sequences in which there are an infinite number of strictly negative terms, or which contain a term $-\infty$, and associate them with the group $\{0\}$. If we make the sequences into a lattice by putting $(m)\leqq(n)$ when $m_r\leqq n_r$ for every r, then the meet (or join) of the characteristic sequences of two groups is the characteristic sequence of their join (or meet). The author finds the condition for two additive groups of rationals to be isomorphic and the condition for one to contain a subgroup isomorphic to the other, in terms of characteristic sequences. *H. A. Thurston* (Bristol).

Steinberg, R. A geometric approach to the representations of the full linear group over a Galois field. Trans. Amer. Math. Soc. **71**, 274–282 (1951).

The paper gives and studies a large family of representations of the full linear group $G=GL(n, q)$ of degree n over the Galois field of q elements which are obtained in parallelism with the representation theory of the symmetric group H on n symbols. If (ν) is a partition $n=\nu_1+\nu_2+\cdots+\nu_n$, each element of G induces a permutation on the set of monotone sequences $e(\nu)$ of ν_1-, $\nu_1+\nu_2$-, $\cdots$, n-dimensional subspaces in the underlying n-dimensional space. Let $\psi(\nu)$ be the character of the permutation representation $C(\nu)$ of G thus obtained. It is shown then that

$$\chi(\nu)=\sum_{\kappa}\operatorname{sgn}(\kappa_i)\psi(\nu_i+i-1-\kappa_i)$$

is an irreducible character, where the summation is over $n!$ permutations $(\kappa_1, \cdots, \kappa_n)$ of $(0, \cdots, n-1)$ and ψ is defined to be 0 if any argument is negative. The proof consists in considering a Kronecker product of two C's, bringing it into correspondence with a similar product representation of H, by a geometrical lemma, applying the orthogonality of characters, and leaning on the character theory of H. The χ corresponding to different partitions are different. The degrees of χ are determined. $\chi(\nu)$ with $(\nu)=(0^{n-1}n)$ is explicitly computed. Further, the family of characters derived from the χ's of smaller full linear groups by multiplication of determinants, multiplication, and induction, is considered. The paper closes with a conjecture on the generating function of the numbers of different degrees of characters.

T. Nakayama (Nagoya).

***Springer, Tonny Albert. Over Symplectische Transformaties. [On Symplectic Transformations].** Thesis, University of Leiden, 1951. 35 pp. (Dutch. French summary)

Symplectic transformations in n variables with a commutative coefficient field whose characteristic differs from 2 are investigated and invariants are established for the classes of conjugate elements. As in the case of the group of all non-singular linear transformations these invariants consist of irreducible polynomials and systems of non-negative integral numbers, but apart from these also equivalence classes of hermitian forms and of quadratic forms have to be included. Special cases had previously been examined by L. E. Dickson [Trans. Amer. Math. Soc. **2**, 103–138 (1901); Amer. J. Math. **26**, 243–318 (1904)] and J. Williamson [Amer. J. Math. **59**, 599–617 (1937)]. The structure of the normalizer of an arbitrary element of a symplectic group is studied and results analogous to those of Dieudonné [Bull. Soc. Math. France **74**, 130–146 (1946); these Rev. **9**, 264] for the general linear group are found.

O. Taussky-Todd (Washington, D. C.).

Peremans, W. Finite binary projective groups. Compositio Math. **9**, 97–129 (1951).

The binary projective group over a field K can be represented as the group of all fractional linear transformations (1) $z'=(az+b)/(cz+d)$ $(ad-bc\neq 0)$, where a, b, c, d lie in K. When K is the field of complex numbers, a classical result states that there exist only five types of finite groups of such transformations, namely (A) cyclic groups, (B) dihedral groups, (C) the tetrahedral group $\mathfrak{A}_4$, (D) the octahedral group $\mathfrak{S}_4$ and (E) the icosahedral group $\mathfrak{A}_5$.

In the present paper the search for finite binary projective groups is extended to the case of an arbitrary ground field K. If K has a non-zero characteristic p, new features arise for the following reason: the fixed points or poles of a transformation are the roots of the quadratic equation obtained by putting $z'=z$ in (1). A transformation with two coincident poles is called parabolic, and if the double pole is taken to be at ∞, formula (1) reduces to (2) $z'=z+b$. If $b\neq 0$, such a transformation cannot be an element of a finite group, unless K has a non-zero characteristic p, when the transformation (2) is of order p. The set of all these transformations forms what is called an "additive" group, whose order is a power of p. It is the presence of parabolic elements which brings about a richer harvest of finite binary projective groups over a general field K. Besides the five classical groups of non-parabolic elements there exist seven possible types of finite groups with parabolic elements. Some of these new groups are isomorphic with groups listed under (A) to (E), but they also include a metacyclic group of order dp^m, where $d \mid p^m-1$ and the groups $PGL(2, p^m)$ and $PSL(2, p^m)$. Certain restrictions must be imposed on p in order that the groups in question may exist. For their actual constructions the author assumes in the present paper that the ground field has been suitably extended. An interesting result is the fact that any two groups of the same type are conjugate in the full projective group except in the case of the additive and the metacyclic groups. This is proved by constructing a canonical form for the generators of each group in the extended field.

The author's treatment of the problem is lucid and very thorough. The paper includes a list of all conjugate cyclic and additive subgroups of each group under discussion.

W. Ledermann (Manchester).

Peremans, W. Existence and equivalence of finite binary projective groups. Compositio Math. **9**, 169–192 (1951).

This is a continuation of the author's paper reviewed above, where it was shown that all possible types of finite binary projective groups can be realized provided the ground field K is suitably extended. The author now considers the question of existence in the original field K when no extensions are allowed. As before, the treatment differs, according as the group does or does not contain parabolic transformations. Necessary and sufficient conditions are derived for the existence of each type of group in K. For example, a non-parabolic transformation of order N exists if and only if 1) N is not divisible by the characteristic and 2) $\zeta+\zeta^{-1}$ lies in K where ζ denotes a primitive Nth root of unity. One of the conditions for the existence of $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$, without parabolic elements, is that K should be a splitting field of the quaternions. This leads to a digression on the problem as to what fields are splitting fields of the quaternions.

In the second part of the paper the author establishes conditions under which two groups of the same type are conjugate in the full projective group over K, when no algebraic extensions are allowed. *W. Ledermann.*

***Gleason, A. M. One-parameter subgroups and Hilbert's fifth problem.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 451–452. Amer. Math. Soc., Providence, R. I., 1952.

This is an abstract of the author's address delivered at the International Congress of Mathematicians, 1950. The author sketches a possible approach to the solution of Hilbert's fifth problem, emphasizing the importance of one-parameter (local) subgroups in a locally Euclidean group G. It was proved by Chevalley and independently by the author that if G contains no arbitrarily small subgroup, then a suitable neighborhood of the identity of G is covered canonically by a family of local one-parameter subgroups. Therefore Hilbert's fifth problem can be divided into two parts: 1) to prove the non-existence of arbitrarily small subgroups in a locally Euclidean group G; and 2) to prove that G is a Lie group, using the fact that a neighborhood of the identity is covered by one-parameter subgroups. The author makes several remarks on these problems, stating that the second problem may be more difficult to solve than the first one.

K. Iwasawa (Princeton, N. J.).

Hochschild, G. Group extensions of Lie groups. II. Ann. of Math. (2) **54**, 537–551 (1951).

In a previous paper, the author began to study group extensions of Lie groups, mainly discussing connected abelian Lie groups [Ann. of Math. (2) **54**, 96–109 (1951); these Rev. **13**, 12]. Here, group extensions of arbitrary Lie groups are considered. First, a theorem of Eilenberg and MacLane is extended to topological groups, so that the enumeration of the extensions of an arbitrary topological group Q by another topological group H can be reduced to the enumeration of the extensions of the center C of Q by H. This reduces the study of group extensions of arbitrary Lie groups to the case where the kernel Q is an abelian Lie group. Then, the relations between the extensions, by a fixed connected Lie group H, of an abelian Lie group C, of its connected component of the identity C_1, and of the discrete factor group C/C_1 are studied and the following results are obtained. Let Ext (C, H, ψ) denote the group of equivalent classes of extensions of C by H, which induce the same homomorphism ψ of H into the group of automorphisms of C, and let Ext (C_1, H, ψ_1) and Ext $(C/C_1, H)$ be similar groups for C_1 and C/C_1 respectively, where ψ_1 is the homomorphism induced by ψ in a natural manner. Then, a canonical isomorphism λ of Ext (C_1, H_1, ψ_1) into Ext (C, H, ψ) as well as a canonical homomorphism μ of Ext (C, H, ψ) into Ext $(C/C_1, H)$ are defined, such that the image of λ coincides with the kernel of μ. If, in particular, ψ is the trivial mapping, the above scheme resolves into the simple statement that Ext (C, H) is the direct product of Ext (C_1, H_1) and Ext $(C/C_1, H)$. The analogous result holds for arbitrary ψ whenever H is semi-simple, and also when H is compact and every element of C/C_1 is of finite order. On the other hand, the group Ext $(C/C_1, H)$ is proved to be isomorphic to the group of all homomorphisms of the fundamental group of H into C/C_1, while the group Ext (C_1, H_1, ψ_1) was studied in detail in the previous paper. Therefore, the structure of Ext (C, H, ψ) can be given precisely in those cases as stated above.

K. Iwasawa (Princeton, N. J.).

de Siebenthal, Jean. Sur les sous-groupes fermés connexes d'un groupe de Lie clos. Comment. Math. Helv. 25, 210–256 (1951).

Let G be a connected and compact Lie group of rank l. According to Cartan, Weil and Stiefel [cf. Stiefel, Comment. Math. Helv. 14, 350–380 (1942); these Rev. 4, 134], there are certain linear forms, called angular parameters of G, defined over the universal covering space R^l of the maximal toral subgroup of G. Suppose G_1 to be a closed subgroup of rank k. Since the maximal toral subgroups of G are conjugate to one another, the space R^k over which the angular parameters of G_1 are defined can be regarded as a subspace of R^l. It is shown that given an angular parameter ρ of G_1 there exist angular parameters $\omega_1, \omega_2, \cdots$ of G such that each ω restricted to R^k coincides with ρ. These ω's are said to be associated with ρ. The author introduces and discusses the following table $I(G, G_1)$:

$$\begin{array}{ll} \rho_1: & \alpha_1, \alpha_2, \cdots, \alpha_{n_1} \\ \rho_2: & \beta_1, \beta_2, \cdots, \beta_{n_2} \\ \cdots & \cdots\cdots\cdots \\ \rho_k: & \gamma_1, \gamma_2, \cdots, \gamma_{n_k} \end{array}$$

where $\{\rho_1, \rho_2, \cdots, \rho_k\}$ is a set of fundamental angular parameters of G_1, the α's are the angular parameters of G associated with ρ_1, the β's are those associated with ρ_2, etc. In case G has no proper subgroup of rank l containing G_1, it is proved, among other things, that (1) the center of G_1 is the intersection of G_1 with the center of G, and (2) if G is semi-simple, then each angular parameter of G is a linear combination of the parameters in the table $I(G, G_1)$ with non-negative integer coefficients. The author calls a closed subgroup G_1 of G an (H_0)-subgroup if the parameters in the table $I(G, G_1)$ form a fundamental set of angular parameters of G. The (H_0)-subgroups of rank $\geqq 2$ of a simple Lie group are completely determined, and the (H_0)-subgroups of rank 1 are discussed. *H. C. Wang* (Princeton, N. J.).

Borel, Armand. Sur la cohomologie des espaces homogènes des groupes de Lie compacts. C. R. Acad. Sci. Paris **233**, 569–571 (1951).

Several theorems are announced on the cohomology of homogeneous spaces with a connected compact Lie group by the application of the spectral cohomology theory. If U is a closed subgroup of a connected compact Lie group G, the characteristic subring of the cohomology ring $H(G/U, A)$ over the coefficient ring A is understood to be the ring of characteristic classes in the base space G/U of the principal fibre bundle G with fibres which are homeomorphic to U. If T is a maximal toroid of G and if G and G/T are both without p-torsion, then $H(G/T, Z_p)$ is equal to its characteristic subring. Under these hypotheses the canonical homomorphism $H(B_G, Z_p) \to H(B_T, Z_p)$, where B_G, B_T are the base spaces of the universal bundles with the groups G, T respectively, is an onto-isomorphism, the image being $I_G \otimes Z_p$, where I_G is the ring of the elements of $H(B_T, Z)$ invariant under the operations of N/T (N = normalisator of T in G). Let S be a maximal toroid of the subgroup U. If U and U/S are without p-torsion, U/S is totally non-homologous to zero with coefficients Z_p in G/S, and the spaces G/S and G/U are simultaneously with or without torsion. If U and G have the same rank and if G/S and G are without p-torsion, then $H(G/U, Z_p)$ is equal to its characteristic subring. Similar statements are valid with rational coefficients. Finally, a generalization is given to coefficients mod p of Hopf's theorem on Γ-manifolds. No proofs are given.

S. Chern (Chicago, Ill.).

Borel, Armand, et Serre, Jean-Pierre. Détermination des p-puissances réduites de Steenrod dans la cohomologie des groupes classiques. Applications. C. R. Acad. Sci. Paris **233**, 680–682 (1951).

Generalizing his squaring operations, Steenrod introduced in 1950 the reduced pth powers, which are homomorphisms $\mathrm{St}_p^i: H^q(X, Z_p) \to H^{q+i}(X, Z_p)$, where X is a polyhedron, p a prime, and $0 \leqq i \leqq q(p-1)$. (The results have been available in the form of mimeographed notes, but have not appeared in print.) The authors devise methods to compute these homomorphisms for certain Lie groups. In particular, if $P_1, \cdots, P_n$ denote the absolutely transgressive elements of degrees $1, 3, \cdots, 2n-1$, respectively, which generate the cohomology algebra $H(U(n))$ of the unitary group $U(n)$ in n variables, the authors find

$$\mathrm{St}_p^{2k(p-1)}(P_i) = (-1)^{i+k+1}\left(\frac{p}{2}\right)! b_p^{k,i} P_j, \quad j = i + k(p-1),$$

where $b_p^{k,i}$ are constants. This formula implies that the 2-sphere and the 6-sphere are the only spheres admitting an almost complex structure. Several other results are announced on the non-existence of cross-sections of certain fiber bundles. The note is rich in content and ideas and, among other things, definitely establishes the reduced pth powers to be an important operation in algebraic topology.

S. Chern (Chicago, Ill.).

Vilenkin, N. Ya. The theory of characters of topological Abelian groups with boundedness given. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 439–462 (1951). (Russian)

This investigates the theory of character groups of an abelian general topological group G in which there is imposed a system of "bounded" sets, as defined by Hu [J. Math. Pures Appl. **28**, 287–320 (1949); these Rev. **11**, 452]. A class of subsets of G is a basis for this "boundedness" if every bounded subset of G is a subset of at least one of the members of the basis. The subset A of G is called quasiconvex if for every element g of G which is not in A there exists a character χ such that $|\chi(g)| > \frac{1}{2}$ but $|\chi(a)| \leqq \frac{1}{2}$ for every $a \varepsilon A$. If each set of a basis is replaced by its quasiconvex hull and if each neighborhood of a neighborhood-basis of the identity of G is replaced by its quasiconvex hull, one derives from G a generalized topological group G^*. Let H denote the closure of the identity-element of G^*. Then the factor-group G^*/H is called the reduced-group of G. It is shown to be locally quasiconvex and to possess a "quasiconvex boundedness." It is called a group of type Q, or more simply, a Q-group. The group of characters of G and of its reduced group are "completely" isomorphic. The class of Q-groups includes all locally bicompact abelian groups. The bounded sets A of G furnish the identity-neighborhoods $N(A)$ of the character group X of G: $N(A) = \{\chi \varepsilon X, \chi(a) \leqq \frac{1}{2}, a \varepsilon A\}$; and the neighborhoods of e in G form a basis for boundedness in X.

The author studies involutory groups (groups isomorphic to the character group of their character group), and gives a sufficient condition that a closed subgroup of an involutory group be itself involutory. Other topics include direct-sums, factor groups, various types of boundedness of sets and of groups, and in particular bounded sequences.

L. Zippin (Brooklyn, N. Y.).

Montgomery, Deane. Locally homogeneous spaces. Ann. of Math. (2) **52**, 261–271 (1950).

Ein metrischer Raum R wird "lokal homogen" genannt, wenn jeder Punkt p eine Umgebung $U(p)$ besitzt, derart

dass es für je zwei (hinreichend benachbarte) Punkte $a \varepsilon U(p)$, $b \varepsilon R$ eine beliebig kleine Deformation von $U(p)$ gibt, die a in b überführt; dabei ist unter einer Deformation eine stetige 1-parametrige Familie von Homöomorphismen von $U(p)$ in R zu verstehen, die mit der Identität beginnt. Jede Mannigfaltigkeit ist lokal homogen; umgekehrt wird gezeigt, dass ein lokal homogener separabler metrischer Raum M von endlicher Dimension n viele der Zusammenhangs- und Trennungseigenschaften aufweist, die für n-dimensionale Mannigfaltigkeiten gelten. Es werden u.a. folgende Eigenschaften von M hergeleitet: (1) Eine abgeschlossene n-dimensionale Teilmenge von M hat innere Punkte. (2) Je zwei Punkte von M haben homöomorphe Umgebungen. (3) M ist in der Dimension $n-1$ lokal zusammenhängend (im Homologiesinne). (4) Jede $(n-1)$-dimensionale abgeschlossene Teilmenge N von M ist in einer zusammenhängenden offenen Teilmenge enthalten, welche durch N zerlegt wird. (5) Wird über die Deformationsfamilien der $U(p)$ eine etwas stärkere Voraussetzung gemacht, so ist M auch in der Dimension $n-2$ lokal zusammenhängend (im Homologiesinne). (6) Im Falle $n=2$ ist M eine Mannigfaltigkeit. Jeder n-dimensionale Wirkungsraum einer lokal zusammenhängenden lokal kompakten Gruppe ist lokal homogen und hat somit die in der Arbeit für M hergeleiteten Eigenschaften. Ein Anhang handelt von beliebig kleinen endlichen Untergruppen einer n-dimensionalen topologischen Gruppe der genannten Art; es wird eine lokale Voraussetzung formuliert, unter welcher es (ausser e) keine solchen Untergruppen gibt. *B. Eckmann* (Urbana, Ill.).

Britton, J. L., and Shepperd, J. A. H. Almost ordered groups. Proc. London Math. Soc. (3) **1**, 188–199 (1951).

Let G be a group which is also a partly ordered set under a relation $<$. We call G a partly ordered group (po-group) if the following condition holds. Axiom po: If $a<b$ then $ag<bg$ and $ga<gb$ for all g in G. We call G an almost ordered group (ao-group) if the following condition holds. Axiom ao: If $a<b$ then $ag<bg$ for most elements g in G and $ha<hb$ for most elements h in G. Here the set property, "S contains most elements of G", can have any interpretation provided that (i) if S_1 and S_2 each contain most elements of G then $S_1 \cap S_2$ is not empty, (ii) if S contains most elements of G then so do gS and Sg for all g in G, and (iii) If S_1 contains most elements in G and if $S_1 \subseteq S_2$ then S_2 contains most elements in G. We call a po-group G an almost fully ordered group (afo-group) if the following condition holds. Axiom afo: There are at most a finite number of elements that are incomparable with the identity element. Every ao-group can be made into a po-group by an extension of $<$; two different elements of finite order in an ao-group are incomparable. In an afo-group there are only a finite number of elements of finite order; these constitute a normal subgroup F and there exists an extension of $<$ in which the set of elements incomparable with or equal to the identity element is exactly F. The factor group G/F is fully ordered under the induced order relation. *R. M. Thrall.*

Iseki, Kiyoshi. On simply ordered groups. Portugaliae Math. **10**, 86–88 (1951).

A group G whose elements are simply ordered and such that if $a<b$ then $ac<bc$ and $ca<cb$ for all c in G is called an ordered group. Let (P, Q) be a Dedekind cut of G for which a in P and b in Q imply $a<b$. The cut (P, Q) is called continuous if P has a maximum or Q has a minimum but not both. (P, Q) is called a lacuna if P has no maximum and Q has no minimum. Using Montgomery's result [Ann. of Math. (2) **49**, 110–117 (1948); these Rev. **9**, 496] it is shown that if every cut is continuous then G is isomorphic to the additive group of real numbers. Results of C. J. Everett [Duke Math. J. **11**, 109–119 (1944); these Rev. **5**, 169] and of Everett and S. Ulam [Trans. Amer. Math. Soc. **57**, 208–216 (1945); these Rev. **7**, 4] are used to prove the known theorem that any archimedean ordered group is isomorphic to a subgroup of the additive group of real numbers. It is stated that if G is ordered and dense in itself then each of (1) G has the Lebesgue-Khintchine property, (2) no cut of G is a lacuna, (3) every non-void subset bounded above (below) has a supremum (infimum) implies that G is isomorphic to the additive group of real numbers.

L. W. Cohen (Flushing, N. Y.).

Michiura, Tadashi. On simply ordered groups. Portugaliae Math. **10**, 89–95 (1951).

Let G be an ordered group. A positive element $a \varepsilon G$ is non-archimedean if the positive powers of a are bounded in G. A normal subgroup H of G is called an l-ideal [cf. G. Birkhoff, Lattice theory, rev. ed., Amer. Math. Soc. Colloq. Publ., v. 25, New York, 1948; these Rev. **10**, 673] if $a_1 \leqq b \leqq a_2$ and $a_1, a_2 \varepsilon H$ imply $b \varepsilon H$. The results concern groups G which satisfy the condition that the cyclic subgroups generated by non-archimedean elements of G are normal. The results are: (1) An l-ideal M of G is maximal if and only if the ordered factor group G/M is archimedean. (2) G is isomorphic to a subgroup of the lexicographically ordered n-dimensional real vector group if G has exactly $n+1$ l-ideals. (3) The groups G are commutative.

L. W. Cohen (Flushing, N. Y.).

NUMBER THEORY

Beeger, N. G. W. H. On even numbers m dividing 2^m-2. Amer. Math. Monthly **58**, 553–555 (1951).

The author proves that there are infinitely many even numbers m satisfying $2^m \equiv 2 \pmod{m}$. The first such number was recently found by Lehmer, $m = 2 \times 73 \times 1103$.

P. Erdös (Aberdeen).

Popadić, Milan S. Generalisation of a problem of J. Karamata on a kind of sequences. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **3**, no. 7, 13 pp. (1950). (Serbo-Croatian. English summary)

An elementary discussion of the number of integers in $[1, x]$ which contain a given factor k an even number of times. *G. G. Lorentz* (Toronto, Ont.).

Fabo, Kiril. On the factorization of large numbers. Scienca Revuo **1**, 49–57 (1949). (Esperanto)

Several methods for the factorization of large numbers are discussed. These include the following. (I) Digital divisibility tests for small trial divisors. (II) The use of Euclid's algorithm. The greatest common divisor of the given number and the product of small primes reveals any small prime factors of the given number. Incidentally, the product of the primes between 13 and 101 inclusive is given incorrectly and should be

1 00806 21833 70118 44545 48195 53595 26997.

(III) The use of the equation $a^2-b^2=(a+b)(a-b)$. The problem of representing a given number N as the difference

of two squares may be facilitated by the use of small moduli and their quadratic residue. Two practical aids to this method are described: (a) movable paper strips, (b) mechanical sieves. *M. Klein* (Tel Aviv).

Bastien, L. Remark on the article of Kiril Fabo, "On the factorization of large numbers". Scienca Revuo 2, 102–103 (1950). (Esperanto)

Historical comments about Fermat. *D. H. Lehmer.*

Fabo, Kiril. On prime numbers. Scienca Revuo 1, 126–134 (1949). (Esperanto)

An expository article giving data on the distribution of primes and tests for primality. *D. H. Lehmer.*

Richert, Hans-Egon. On permutable prime numbers. Norsk Mat. Tidsskr. 33, 50–53 (1951). (Norwegian)

A permutable prime number is a positive integer which is a prime for every permutation of the digits to base 10. It is established that, apart from primes of the form $111\cdots1$, there are no permutable primes with n digits for $3<n<6\cdot10^{175}$. *I. Niven* (Eugene, Ore.).

Niven, Ivan. Functions which represent prime numbers. Proc. Amer. Math. Soc. 2, 753–755 (1951).

W. H. Mills [Bull. Amer. Math. Soc. 53, 604 (1947); these Rev. 8, 567] showed that there exists a number $A>1$ such that $[A^{3^n}]$ is a prime for every positive integer n. The author shows that 3 can be replaced by any real number $c>8/3$. (For $c=3, 4, 5, \cdots$ this was proved by L. Kuipers [Nederl. Akad. Wetensch., Proc. 53, 309–310 = Indagationes Math. 12, 57–58 (1950); these Rev. 11, 644].) Furthermore he proves that, given any $A>1$, there exists a number c such that $[A^{c^n}]$ is a prime for $n=1, 2, 3, \cdots$.

N. G. de Bruijn (Delft).

Wright, E. M. A prime-representing function. Amer. Math. Monthly 58, 616–618 (1951).

Let $[y]$ denote the greatest integer less than or equal to y. It is shown that there is a real number α such that if $g_0=\alpha$ and $g_{n+1}=2^{g_n}$ for $n\geqq0$, then $[g_n]$ is a prime for all positive integral n. The proof uses only the elementary result that if $N\geqq2$, there is a prime between N and $2N$.

W. H. Mills (New Haven, Conn.).

Kuipers, L. On the representation of integers by sums of polynomials. Proc. Amer. Math. Soc. 2, 750–752 (1951).

Let $\Delta f(x)=f(x+1)-f(x)$, $\Delta^i f=\Delta(\Delta^{i-1}f)$ and

$$F_k(x)=x(x-1)\cdots(x-k+1)/k!.$$

Then $\Delta F_k(x)=F_{k-1}(x)$. Let $f(x)=a_kF_k(x)+\cdots+a_1F_1(x)+a_0$. Then $\Delta^{k-1}f(x)=a_kx+a_{k-1}$. Let $H(f(x))$ be the least integer s such that every integer is a sum or difference of s values of $f(x)$ for integral x, and let $\Gamma(f(x))$ be the least integer t such that every integer is congruent to a sum or difference of t values of $f(x)$ mod a_k. By the above calculation we have evidently $H(f(x))\leqq2^k+\Gamma(f(x))$. From this principle the author deduces: (i) Every integer can be expressed in infinitely many ways as a sum of 4^k+1 values of the $(2k+1)$-th degree polynomial

$$a(x+k)(x+k-1)\cdots(x-k)/(2k+1)!+gx+c,$$

where $(a, g)=1$, and (ii) if $a=1$, we can replace 4^k+1 by 4^k in (i). In the statement of theorems I and II read $2k+1$ for 2^k+1. *L. K. Hua* (Peking).

Dubisch, Roy. Representation of the integers by positive integers. Amer. Math. Monthly 58, 615–616 (1951).

Shanks, Daniel. A short proof of an identity of Euler. Proc. Amer. Math. Soc. 2, 747–749 (1951).

Let $P_n=\prod_{r=1}^{n}(1-x^r)$, $Q_n=\prod_{r=1}^{n}(1-x^{2r})/(1-x^{2r-1})$. Then

$$(1)\quad \sum_{s=0}^{n}(-1)^s\frac{P_n}{P_s}x^{s+3s(s+1)/2}=1+\sum_{s=1}^{n}(-1)^s[x^{s(3s-1)/2}+x^{s(3s+1)/2}],$$

$$(2)\quad \sum_{s=0}^{n}\frac{Q_n}{Q_s}x^{s(3s+1)}=1+\sum_{s=1}^{n}[x^{s(3s-1)}+x^{s(3s+1)}].$$

These finite identities, discovered by application of a non-linear transformation to certain sequences, appear to be new. For $n\to\infty$, (1) and (2) yield well-known identities of Euler and Gauss, respectively. *N. J. Fine.*

Turán, Paul. A note on Fermat's conjecture. J. Indian Math. Soc. (N.S.) 15, 47–50 (1951).

Let $R_q(N)$ denote the number of solutions of $x^q+y^q=z^q$ in integers x, y, z with $1\leqq x\leqq N$, $1\leqq y\leqq N$, $1\leqq z\leqq N$ where x and y are coprime and q is an odd prime. According to Fermat's last theorem $R_q(N)=0$ for all $N\geqq1$. The author shows that there is a c_1 depending on q alone such that $R_q(N)<c_1N(\log N)^k$ where $k=2+(q-1)^{-1}$. This result is made to follow from the form of the prime factors of x^q+y^q and from the prime number theorem for primes in arithmetical progression. *D. H. Lehmer.*

Oblath, R. Une équation diophantienne de M. Segre (addition). Bull. Soc. Roy. Sci. Liège 20, 378 (1951).

A correction to the author's paper [same Bull. 20, 199–204 (1951); these Rev. 13, 208]. *J. W. S. Cassels.*

Aucoin, A. A. Systems of Diophantine equations. Proc. Amer. Math. Soc. 2, 760–765 (1951).

Three theorems on diophantine equations of which the most impressive is that if $L_{ij}(x)$ $(i, j=1, \cdots, n)$ are n^2 homogeneous linear forms in q variables $(x)=(x_1, \cdots, x_q)$ with integer coefficients and if $f_i(y)$ are homogeneous polynomials of degree m prime to n then in some cases a general solution of (*) $\prod_j L_{ij}(x)=f_i(y)$ can be given. The author assumes (tacitly) that the n^2-n forms L_{ij} $(j\neq n)$ are linearly dependent on at most $q-n$ of them. If these and the (y) are taken as parameters, then (*) become n linear equations in (x). Thus we have $(q-n)+n=q$ linear equations in (x); which can be solved, the solution being rational if the parameters are. Finally, integer solutions are obtained by homogeneity. *J. W. S. Cassels* (Cambridge, England).

Skolem, Th. Remarks concerning the indeterminate equation $xy+yz+zx=k$, k a positive integer, together with the analogue in several unknowns. Norsk Mat. Tidsskr. 33, 41–49 (1951). (Norwegian)

By use of the transformation $x'=x+2z$, $y'=y+2z$, $z'=-z$ which leaves the form $xy+xz+yz$ invariant, the author obtains a "least" solution from any known solution of the equation in the title. This least solution has $x\geqq y\geqq z\geqq0$ and $z\leqq(k/3)^{1/2}$. The extension to n unknowns is discussed.

I. Niven (Eugene, Ore.).

Apostol, T. M., and Zuckerman, Herbert S. On magic squares constructed by the uniform step method. Proc. Amer. Math. Soc. 2, 557–565 (1951).

The theory of the construction of magic squares by the "uniform step process" was given by D. N. Lehmer [Trans. Amer. Math. Soc. 31, 529–551 (1929)]. If n^2 is the number of cells in the square then the coordinates (A_s, B_s) of the

cell containing the number x ($x=1(1)n^2$) are given by the formulas

$$A_x \equiv p+\alpha(x-1)+a\left[\frac{x-1}{n}\right] \pmod n,$$

$$B_x \equiv q+\beta(x-1)+b\left[\frac{x-1}{n}\right] \pmod n,$$

where (p, q) is the cell in which the number 1 is located, (α, β) is the "step" and (a, b) is the "break step." The numbers α, β, a, b and their determinant $\Delta=\alpha b-a\beta$ determine the nature of the square. If all five numbers are prime to n, then n is odd and the square is magic in rows and columns. The present paper is concerned with magic squares in which this condition is not necessarily satisfied. Formulas for (A_x, B_x) of the type

$$A_x \equiv p+\alpha(x-1)+a\left[\frac{(x-1)\delta}{n}\right]+c\left[\frac{(x-1)\delta\epsilon\theta}{n^2}\right] \pmod n,$$

$$B_x \equiv q+\beta(x-1)+b\left[\frac{(x-1)\delta}{n}\right]+d\left[\frac{(x-1)\delta\epsilon\theta}{n^2}\right] \pmod n$$

are considered in which a second break step (c, d) is introduced. The integers δ, ϵ, are the greatest common divisors: $\delta=(\alpha, n)=(\beta, n)$, $\epsilon=(a, n)=(b, n)$ while $\theta=(\Delta, n)\delta^{-1}\epsilon^{-1}$. Conditions for the square to be filled, for it to be magic in rows and columns, to be symmetric, and to be diabolic are obtained in the case of these general squares.

D. H. Lehmer (Los Angeles, Calif.).

Schnee, Walter. Über magische Quadrate und lineare Gitterpunktprobleme. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **98**, no. 1, 47 pp. (1951).

The author considers a semi-magic square formed by nine different integers, not necessarily consecutive, such that the sum of the three in any row or column has the same value s. He shows that the obvious transformations suffice to convert this into a "normal square" in which the smallest number is 1 and occurs in the top left corner while the greatest occurs in the top right corner and the next greatest in the bottom left; e.g.

1	5	9	1	6	12
6	7	2	7	8	4
8	3	4	11	5	3
($s=15$)			($s=19$)		

He defines $f(s)$ to be the number of normal squares for a given sum s; e.g., $f(s)=0$ for $s<15$, $f(15)=1$, $f(16)=2$, $f(17)=4$. He shows that this is equal to the number of lattice points within a certain tetrahedron but not lying on any of a certain set of seven planes. He deduces an explicit but very complicated formula, one consequence of which is that $f(s)\leqq f(s+1)$ except when $s=12k$ with $k\geqq 5$.

H. S. M. Coxeter (Toronto, Ont.).

Mordell, L. J. Lattice points in a tetrahedron and generalized Dedekind sums. J. Indian Math. Soc. (N.S.) **15**, 41–46 (1951).

Let p, q be two positive integers prime to each other. One form of the reciprocity law for Dedekind sums is given by the formula $s(p, q)+s(q, p)=(pq^{-1}+qp^{-1}+p^{-1}q^{-1}-3)/12$, where $s(p, q)=\sum_{x=1}^{q}x((px/q))/q$ and where $((X))=X-[X]-\frac{1}{2}$ or 0 according as X is not or is an integer. The author's proof of this theorem [Amer. J. Math. **73**, 593–598 (1951); these Rev. **13**, 113] suggests its extension to a set of n positive integers $p, q, r, s, \cdots$ no two of which have common factor. Put $s(p, q, r)=\sum_{x=1}^{r-1}x((pqx/r))/r$ and

$$S_3(p, q, r)=s(p, q, r)+s(q, r, p)+s(r, p, q).$$

Denote by $N_3(p, q, r)$ the number of lattice points in the tetrahedron

$$0\leqq x<p,\ 0\leqq y<q,\ 0\leqq z<r,\ 0<xp^{-1}+yq^{-1}+zr^{-1}<1.$$

Calculating the value of N_3 in two different ways, the author first derives the formula

$$S_3+N_3=\tfrac{1}{6}pqr+\tfrac{1}{4}\textstyle\sum qr+\tfrac{1}{4}\sum p+\tfrac{1}{12}\sum\dfrac{qr}{p}+\dfrac{1}{12pqr}-2,$$

the summations referring to p, q, r. An analogous formula is then obtained for $n=4$. The method of proof shows that for $n>4$, the formula for S_n will depend upon the number of lattice points in sections of an n-dimensional tetrahedron defined by $\lambda<\sum x/p<\lambda+1$ for a number of values of $\lambda\leqq n-1$.

A. L. Whiteman (Los Angeles, Calif.).

Kruyswijk, D. On the number of lattice-points in a wide convex region. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 152–161 (1951).

Let G be a plane, open, convex region with area $I(G)$. Let $A(G)$ be the number of points of the standard lattice (i.e. points with integer coordinates x, y) in G and $a(G)$ the number of lattice points on the boundary of G. The set E consists of all chords k of G each of which contains at least two lattice points in G and cuts off from G an open region ω_k which does not contain lattice points. The region G is defined to be wide if each ω_k of G is contained within an isosceles right angled triangle with hypothenuse k. Let ω_k also denote the area of the region ω_k and k also the length of the chord k. The purpose of the paper is to show that a constant c, independent of G, exists with the property that a chord k exists with

$$|A(G)-I(G)|<c(k^3/\omega_k)^{2/3},\quad a(G)<c(k^3/\omega_k)^{2/3}$$

for every G which is wide. The author proves by induction that $A(P)=I(P)+\frac{1}{2}a(P)-1$ where P is a simple closed polygon whose vertices are lattice points. From this follows easily that $|A(G)-I(G)|<\sum_E a_k+\sum_E \omega_k$ where a_k is one less than the number of lattice points on the chord k. The computations which lead to estimates of these two sums, and hence to the main result, are of an elementary character.

To illustrate the relationship between van der Corput's theorem [Landau, Vorlesungen über Zahlentheorie, v. 2, Hirzel, Leipzig, 1927, pp. 288–289] which connects $I(G)$ and $A(G)$, where G is a region enclosed by the x-axis, two ordinates and a curve $y=f(x)$, and his main result the author uses the latter to obtain, by a simple deduction, the following result which is likewise a consequence of the van der Corput theorem. G is a convex region at each of whose boundary points P a radius of curvature $\rho(P)$ is defined. If r is the least upper bound of $\rho(P)$ then $A(G)=I(G)+O(r^{2/3})$, $a(G)=O(r^{2/3})$, for all regions G homothetic to a given region G_0, the constant used in the O-relationship being independent of G_0.

D. Derry (Vancouver, B. C.).

Mullineux, N. Lattice points in the star body K: $|x_1^3+x_2^3-x_3^3|\leqq 1$, $|x_3|\leqq\sqrt{2}$.

Proc. London Math. Soc. (2) **54**, 1–41 (1951).

Let K be the 3-dimensional star body of points (x_1, x_2, x_3) for which $|x_1^3+x_2^3-x_3^3|\leqq 1$, $|x_3|\leqq\sqrt{2}$. The purpose of

this paper is to determine the critical lattices of K. A construction for these lattices is given. Moreover they are shown to be either the critical lattices of the infinite star body $|x_1^2+x_2^2-x_3^2| \leqq 1$ or lattices which can be transformed into certain of these lattices by an affine transformation: $x_1=x_1'+\tau_1 x_3'$, $x_2=x_2'+\tau_2 x_3'$, $x_3=x_3'$. Their determinant is $\frac{1}{2}\sqrt{6}$. By use of the group of affine transformations which leave the form $x_1^2+x_2^2-x_3^2$ invariant it is shown that critical lattices pass through every boundary point of K.

To prove his results the author considers a certain plane through the origin which contains a two-dimensional sublattice of a given critical lattice of K and which cuts the boundary of K in points on the hyperboloid $x_1^2+x_2^2-x_3^2=1$. Now K is subjected to an affine deformation so that the above plane cuts the deformed K in a circle. Any plane parallel to this plane which cuts the deformation of K will do so in a region bounded by line segments and circular arcs. About half the paper deals with plane lattices without points interior to such regions. These results depend on a study of the critical lattices of symmetric plane regions which consist of the set-theoretic join of the points of a circle and the points interior to two intersecting circles of equal radius.

Results analogous to the main result are stated for the star body of points (x_1, x_2, x_3) for which $-2 \leqq x_1^2+x_2^2-x_3^2 \leqq 1$, $|x_3| \leqq \sqrt{6}$. *D. Derry* (Vancouver, B. C.).

Cassels, J. W. S., Ledermann, W., and Mahler, K. Farey section in $k(i)$ and $k(\rho)$. Philos. Trans. Roy. Soc. London. Ser. A. **243**, 585–626 (1951).

Let R be the ring of rational integers and k the field of rational numbers. The first type of Farey section in $k(i)$ is $H_N(i)$ the set of all fractions α/β with α, β in $R(i)$ having norms $N(\alpha)$, $N(\beta) \leqq N$. The second type $H_N^*(i)$ does not require $N(\alpha) \leqq N$. For a given N, $R(\alpha, \beta)$ is the set of all complex numbers z for which α/β is the best approximation among the fractions in $H_N(i)$, i.e. for which $|\beta z-\alpha| \leqq |\beta' z-\alpha'|$ for all α'/β' in $H_N(i)$ (similarly, $R^*(\alpha, \beta)$ is defined). Next, α/β, α'/β' are defined to be adjacent if $R(\alpha, \beta)$ and $R(\alpha', \beta')$ have a common point. A median of α/β and α'/β' is a number $(\alpha+\epsilon\alpha')/(\beta+\epsilon\beta')$, where ϵ is a unit, i.e. ± 1 or $\pm i$. The following assertions are proved. Reduced fractions α/β, α'/β' are adjacent if and only if $|\alpha\beta'-\alpha'\beta|=1$ or $2^{\frac{1}{2}}$ and one of their medians is not in $H_N(i)$. Fractions of $H_{N+1}(i)$ not in $H_N(i)$ are medians of adjacent fractions in $H_N(i)$. Similar definitions and results occur for $H_N^*(i)$ and this Farey section is used to prove the known result that if $\alpha, \beta, \gamma, \delta$ are complex numbers, $\alpha\delta-\beta\gamma=1$, then there are ξ, η in $R(i)$, not both 0, such that $|\alpha\xi+\beta\eta| \leqq k^{\frac{1}{2}}$, $|\gamma\xi+\delta\eta| \leqq k^{\frac{1}{2}}$, where $k=2^{\frac{1}{2}}/(3-3^{\frac{1}{2}})$. The regions $R(\alpha, \beta)$ in the complex plane have boundaries which are arcs of circles. Where three or more such regions meet, the set of angles subtended may have only one of three sets of values. Furthermore $R(\alpha, \beta)$ is a star domain about α/β for N sufficiently large, while $R^*(\alpha, \beta)$ is always a star domain. Corresponding results (except those about star domains) are mentioned and proofs indicated for $k(\rho)$, where $\rho^2+\rho+1=0$. Diagrams of $R(\alpha, \beta)$ are given for $H_{10}(i)$, $H_5(i)$, and $H_{19}(\rho)$ and of $R^*(\alpha, \beta)$ for $H^*_{21}(i)$ and $H^*_{19}(\rho)$. *L. Tornheim.*

Rogers, C. A. A note on coverings and packings. J. London Math. Soc. **25**, 327–331 (1950).

Let K be a bounded symmetrical n-dimensional convex body in n-dimensional space and let δ and ϑ be the densities of the closest lattice packing of K into the whole space and of the most economical lattice covering of the whole space by K respectively. Further, δ^* and ϑ^* denote the corresponding quantities when the centres of the bodies are not restricted to form a lattice. Then $\delta \leqq \delta^* \leqq 1 \leqq \vartheta^* \leqq \vartheta$. The author proves that $\vartheta^* \leqq 2^n\delta^*$ and $\vartheta \leqq 3^{n-1}\delta$. The second inequality is an improvement on the result $\vartheta \leqq n^n\delta$ obtained by Hlawka [Monatsh. Math. **53**, 81–131 (1949); these Rev. **11**, 12]. *R. A. Rankin* (Birmingham).

Davenport, Harold. Sur un système de sphères qui recouvrent l'espace à n dimensions. C. R. Acad. Sci. Paris **233**, 571–573 (1951).

If each point of n-dimensional space belongs to at least one of a system of spheres, each of the same volume V, their centres forming a lattice of determinant d, the spheres are said to form a lattice covering of space with density $\vartheta = V/d$. A lattice covering is constructed with $\vartheta < (\frac{1}{2}\pi e+\epsilon_n)^{n/2}$, where $\epsilon_n \to 0$ as $n \to \infty$ (here $(\frac{1}{2}\pi e)^{\frac{1}{2}} = 1.193\cdots$). It is stated that if n is sufficiently large there is a lattice covering with $\vartheta < (1.15)^n$, but that Bambah and the author have proved that $\vartheta > \frac{1}{2}-\epsilon_n$ (where $\epsilon_n \to 0$ as $n \to \infty$) for all lattice coverings. *C. A. Rogers* (London).

Davenport, H. On a principle of Lipschitz. J. London Math. Soc. **26**, 179–183 (1951).

Let R be a closed and bounded region in n-dimensional space. Let $V(R)$ be the volume of R, and $N(R)$ the number of points in R with integral coordinates. The object of this paper is to obtain an upper bound for $|N(R)-V(R)|$ under certain mild conditions on R. Specifically, suppose that any line parallel to one of the coordinate axes intersects R in at most h intervals. Suppose further that the same is true for any of the m-dimensional regions obtained by projecting R on one of the coordinate spaces defined by equating $n-m$ of the coordinates to zero. Let V_m denote the sum of the volumes of these m-dimensional regions if $m \geqq 1$, and put $V_0=1$. Then it is shown that

$$|N(R)-V(R)| \leqq \sum_{m=0}^{n-1} h^{n-m} V_m.$$

W. H. Mills (New Haven, Conn.).

Davenport, H. On the class-number of binary cubic forms. I. J. London Math. Soc. **26**, 183–192 (1951).

An estimate is obtained for the number of classes of properly equivalent irreducible binary cubic forms with integral coefficients and discriminant D satisfying $0 < D \leqq X$. The essential idea is to use Hermite's definition of a reduced cubic form to obtain an approximate one-to-one correspondence between these classes of cubic forms and the points with integral coordinates in a certain four-dimensional region R. The number of such points in R is then estimated by means of the result of the paper reviewed above. In this way it is shown that the number of such classes is $KX+O(X^{15/16})$ as $X \to \infty$, where K is an absolute constant, in fact the volume of a certain four-dimensional region. This constant K is then computed. In formula (22) on page 188, X should be replaced by $3X$. This seems to mean that K should be $\pi^2/36$ instead of $\pi^2/108$. *W. H. Mills.*

Davenport, H. On the class-number of binary cubic forms. II. J. London Math. Soc. **26**, 192–198 (1951).

By methods similar to those used in the preceding paper the corresponding result for cubic forms of negative discriminant is obtained. If $h(D)$ is the number of classes of

properly equivalent irreducible binary cubic forms with integral coefficients and discriminant D, then

$$\sum_{\Delta=1}^{X} h(-\Delta)=\frac{\pi^2}{12}X+O(X^{13/16})$$

as $X\to\infty$. *W. H. Mills* (New Haven, Conn.).

Cassels, J. W. S. A remark on the class number of quadratic forms of given determinant. Proc. Cambridge Philos. Soc. **47**, 820 (1951).

It is proved by induction with respect to n that any quadratic form in n variables with integer coefficients and of a given determinant $D\neq 0$, is equivalent to at least one of a finite set $\mathfrak{S}(n, D)$ of forms. The author points out that this is a very simple proof of the finiteness of the class number of such forms. *C. Arf* (Istanbul).

Nakayama, Tadasi. Factor system approach to the isomorphism and reciprocity theorems. J. Math. Soc. Japan **3**, 52–58 (1951).

Let K be a finite Abelian extension of an algebraic number field k, with Galois group $\mathfrak{G}$. Let $\mathfrak{C}_K$ and $\mathfrak{C}_k$ be the idèle-class groups of K and k respectively. A canonical factor system $\mathfrak{a}$ in $H^2(\mathfrak{G}, \mathfrak{C}_K)$ is constructed with the property that $\sigma\to\prod_{\tau\in\mathfrak{G}}\mathfrak{a}(\sigma, \tau)$ induces the reciprocity law isomorphism between $\mathfrak{G}$ and the norm-class idèle-class group $\mathfrak{C}_k/N_{K/k}\mathfrak{C}_K$. This canonical factor system can be used in the proof of the reciprocity law. *W. H. Mills.*

Dénes, Peter. Über eine rekurrente Serie von relativ-zyklischen algebraischen Zahlkörpern. Monatsh. Math. **55**, 229–232 (1951).

Let K_0 denote the field of the pth roots of unity over the rationals, p being an odd prime. The author considers a series of Kummer extensions $K_{i+1}=K_i(M_i^{1/p})$, where M_i is a number in K_i which is not a pth power in K_i. Apart from results on the prime divisors of the discriminant and different of K_i which are almost immediate consequences of the known decomposition laws, he obtains a generalization of a result of Pollaczek [Math. Z. **21**, 1–38 (1924)], which is the case $i=1$, to the effect that if each M_j $(j<i)$ is a non-primary unit (i.e., not congruent mod $(1-\varphi)^p$ to the pth power of a number in K_j, where φ is a primitive pth root of unity) then the class number of K_i is not divisible by p. By a result of Vandiver [Bull. Amer. Math. Soc. **40**, 855–858 (1934)], this implies that every primary unit of K_i is a pth power in K_i. *G. Hochschild* (Urbana, Ill.).

Gut, Max. Kubische Klassenkörper über quadratisch-imaginären Grundkörpern. Nieuw Arch. Wiskunde (2) **23**, 185–189 (1951).

The author constructs explicitly infinitely many imaginary quadratic fields whose class numbers are $\equiv 0 \pmod 3$, and, for each one of these fields, an unramified cyclic extension of degree 3. These fields are obtained as follows. Let an integer q be of the form $81a+25$, where a is a rational integer >0; then the splitting field of the equation $x^3-3x-q=0$ is of degree 6, and is unramified over its quadratic subfield, which is imaginary. *C. Chevalley.*

Hasse, Helmut. Zur Geschlechtertheorie in quadratischen Zahlkörpern. J. Math. Soc. Japan **3**, 45–51 (1951).

The author develops in this note the explicit number-theoretic equivalent of Gauss' theory of genera as a consequence of the class field theory for quadratic fields $\Omega=P(\sqrt{d})$ over the rational field P. The approach consists in the exploitation of the mechanism of the law of reciprocity for the maximal unramified extension K/Ω and of the "Geschlechterkörper" Γ, i.e. the maximal abelian extension of P which is unramified over Ω, to the determination of the Galois group of K/P as a group extension of the group of K/Ω where the special arithmetic assumptions on the fields imply the important information that the group of K/Γ (commutator group for K/P) is the square of the group of K/Ω. This fact identifies the class field theoretic structure of Γ/Ω and shows the connection with the classical theory of genera. Next the radicals determining Γ/Ω are put into explicit relation with the primary divisors of the discriminant d which are in turn discriminants of quadratic fields over P. These results are analogous to the bisection of hyperelliptic functions by means of the classical radical functions. Furthermore the quadratic fields corresponding to the primary discriminants are characterized by their class groups in Ω and corresponding characters of the divisor group of Ω whereby some classical results of Hilbert are reformulated. As tool the translation theorem of class field theory figures significantly. In conclusion the author makes more precise one of his earlier results on the principal ideal theorem for complex multiplication for imaginary quadratic fields Ω/P; for the full validity of the principal ideal theorem it is necessary and sufficient that K is disjoint from the field of 12th roots of unity over Ω. The author gives explicit expressions for this condition in terms of norm residue symbols modulo 2 and 3 for the prime divisors of d. *O. F. G. Schilling.*

Hasse, Helmut. Allgemeine Theorie der Gaussschen Summen in algebraischen Zahlkörpern. Abh. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1951**, no. 1, 23 pp. (1951).

Suppose that κ is an algebraic number field with the different $\mathfrak{d}$. Let (1) χ be a congruence character of κ with the conductor $\mathfrak{f}\mathfrak{u}$ ($\mathfrak{u}$ consisting of the infinite prime divisors of the conductor), (2) $\mathfrak{m}\mathfrak{w}$ a multiple of the conductor split into its finite and infinite parts $\mathfrak{m}$ and $\mathfrak{w}$ correspondingly, (3) $\mathfrak{a}$ a divisor of κ with the complement $\bar{\mathfrak{a}}$ defined by $\mathfrak{a}\bar{\mathfrak{a}}=\mathfrak{d}^{-1}$, and (4) n a non-zero element of $\bar{\mathfrak{a}}\mathfrak{m}^{-1}$. Then a general Gaussian sum is defined as $\tau=\tau_n(\chi, \mathfrak{m}\mathfrak{w}|\mathfrak{a})=\sum\chi(\mathfrak{x})e(x\eta)$ where $e(\alpha)$ denotes $e^{2\pi i\beta}$, β the rational trace of α, and where the summation is extended over all residue class representatives x which satisfy $x \bmod \mathfrak{m}\mathfrak{a}$; $x=\mathfrak{x}\mathfrak{a}$, $(\mathfrak{x}, \mathfrak{m})=1$ and $x\equiv 1 \pmod{\mathfrak{w}}$. The author makes a very useful collection of the major formal results concerning the dependency of τ on the entities described above under (1) to (4). The proofs are of elementary nature, that is, they consist essentially in rearrangements of the sum τ based upon the divisor and congruence theory of κ and are parallel to similar processes in the theory of modular functions and L-series. After the ultimate representation of τ as a product of τ's whose defining characters have primary modules the formula for the absolute value of a proper Gaussian sum is found; its proof is modeled after the elementary proof of Rademacher. Finally for the regular ramified case of a prime conductor $\mathfrak{p}$ the sum $\tau_n(\chi|1)=\sum\chi(x)e(x\eta)$, x integral modulo $\mathfrak{p}$, $\eta=d^{-1}\mathfrak{n}\mathfrak{p}^{-1}$, $(\mathfrak{n}, \mathfrak{p})=1$, is rewritten as a Gaussian sum of a finite field and thus a formal relation with the theory of algebraic function fields of one variable over a finite field is established. *O. F. G. Schilling* (Chicago, Ill.).

Kawada, Yukiyosi. On the derivations in number fields. Ann. of Math. (2) **54**, 302–314 (1951).

A characterization of the different of an algebraic number field in terms of derivations of its ring of integers was announced by A. Weil in 1943 [abstract, Bull. Amer. Math.

Soc. 49, 41 (1943)]. In the present paper, the theory of the different is developed completely from the point of view of derivations.

If R is any commutative ring, r a subring of R, and S an R-module, a derivation of R over r into S is an additive homomorphism D of R into S which maps r into (0) and satisfies the identity $D(xy)=xD(y)+yD(x)$. These derivations constitute an R-module denoted $\mathfrak{D}(R, r, S)$. If S is at the same time a ring, such a derivation is called essential if the image $D(R)$ does not consist exclusively of divisors of 0.

Now let k be an algebraic number field, and K a finite algebraic extension of k. Denote by i, I, the rings of integers of k, K, respectively. Let P be a prime ideal in I, and set $p=P\cap i$. Let θ be an element of the ring I_P of the integers of the completion K_P such that $I_P=i_p[\theta]$, and let $f(x)$ be the minimum polynomial of θ with respect to k_p. Let $(f'(\theta))=P^s$. Then it is shown first that $\mathfrak{D}(I_P, i_p, I_P/P^r)$ is isomorphic with I_P/P^r for $r\leqq s$, and with I_P/P^s for $r\geqq s$, and that it contains an essential derivation if and only if $r\leqq s$. Thus, the local different $D(K_P/k_p)$ is characterized as the smallest ideal D $(=P^s)$ for which $\mathfrak{D}(I_P, i_p, I_P/D)$ contains an essential derivation, and, for an arbitrary ideal Q $(=P^r)$, $\mathfrak{D}(I_P, i_p, I_P/Q)$ is isomorphic with $I_P/(Q, D)$. It is then deduced that if e is the ramification index and $ei_p=p^h$ then $e-1\leqq s<(h+1)e$. The multiplicative behavior of the local different for towers of extensions is also derived from the above characterization.

For the global situation, one notes first that

$$\mathfrak{D}(I, i, I/P^r)\approx\mathfrak{D}(I_P, i_p, I_P/P^r),$$

and that, for an arbitrary integral ideal $A=P_1^{r_1}\cdots P_k^{r_k}$, the I-module $\mathfrak{D}(I, i, I/A)$ is isomorphic with the direct sum of the I-modules $\mathfrak{D}(I, i, I/P_j^{r_j})$. From this and the local case it is then shown that the global different $D(K/k)$ (which is equal to the ideal generated by all the $f'(\theta)$ with $\theta\in I$ and $f(x)$ the minimum polynomial of θ with respect to k) can be characterized as the intersection of all ideals A for which $\mathfrak{D}(I, i, I/A)$ contains an essential derivation. At the same time, one finds that the P-component of $D(K/k)$ coincides with $D(K_P/k_p)$, whence one has also the multiplicative behavior of the global different for towers of extensions.

The last part of the paper is devoted to an analysis of the 2-dimensional groups of the cohomology theory in which the derivations constitute the 1-dimensional case. These are the groups of the equivalence classes of the bi-i-linear symmetric functions from $I\times I$ to I/A that satisfy the identity $af(b, c)-f(ab, c)+f(a, bc)-cf(a, b)=0$, two such functions being called equivalent if they differ by a function of the type $f(a, b)=ag(b)-g(ab)+bg(a)$, where g is i-linear. The result is that these groups are isomorphic with $I/(A, D)$ (as I-modules), where $D=D(K/k)$. *G. Hochschild.*

Whaples, George. Local theory of residues. Duke Math. J. 18, 683–688 (1951).

Using the Hasse definition of differentials in a field K of algebraic functions of one variable [H. Hasse, J. Reine Angew. Math. 172, 55–64 (1934)], the proof that the sum of the residues of a differential is 0 proceeds by reduction to the case where K is a rational field. In order to make this reduction, one needs a lemma which expresses a local residue of the trace of a differential (with respect to a subfield) as a trace of a residue (or a sum of traces of residues) of the given differential. The proof of this lemma given by Hasse was rather difficult, and involved in particular a laborious reduction to the case of fields of characteristic 0. The author presents a simpler proof based on reduction (by Galois theory) to cyclic extensions of prime degree, for which a direct verification is possible. Since the author uses the Hasse definition of differentials, he has naturally to assume that the basic field is perfect. *C. Chevalley.*

Iwasawa, Kenkichi, and Tamagawa, Tsuneo. On the group of automorphisms of a function field. J. Math. Soc. Japan 3, 137–147 (1951).

Let K be a field of algebraic functions of one variable over an algebraically closed field of constants k. The genus g of K is assumed to be >0, and its characteristic is denoted by p. The theorem of Hurwitz (extended by F. K. Schmidt to the case where $p>0$) states that, if $g>1$, then the group G of automorphisms of K/k is finite. The authors present a new proof of this theorem based on the study of the representation of G on the space of differentials of the first kind.

Most of the reasoning is concerned with the study of the group $G(P)$ of automorphisms which leave some place P of K fixed. Even if $g=1$, the group $G(P)$ is finite. The authors prove that $G(P)$ has a normal Sylow p-subgroup N of order $\leqq p^2(2g-1)^2$, while $G(P)/N$ is cyclic of order $\leqq 6(2g-1)$ (if $p=0$, then $G(P)$ is cyclic of order $\leqq 6(2g-1)$).

In order to establish this, the authors consider the natural representation ρ of $G(P)$ on the space L of functions of K which are $\equiv 0 \pmod{P^{-(2g+1)}}$; ρ is faithful, because $k(L)=K$. The space L has a base $\{x_1, \cdots, x_{r-1}, x_r\}$ such that the divisor of poles of x_i is $P^{-h(i)}$ and

$$h(1)>h(2)>\cdots>h(r-1)>h(r)=0.$$

The space L_i spanned by $x_i, \cdots, x_r$ is mapped into itself by the operations of $\rho(G(P))$. We have $\rho(s)x_i\equiv(c(s))^{-h(i)}x_i \pmod{L_{i+1}}$, where $c(s)\in k$ depends only on s. The mapping $s\to c(s)$ is a representation of $G(P)$ in the multiplicative group of constants $\neq 0$; let N be its kernel. If $s\in N$, then we have $\rho(s)x_{r-1}=x_{r-1}+a(s)$, and $s\to a(s)$ is a representation of N into the additive group of k; let N_1 be its kernel. By Galois theory, N_1 is finite of order $\leqq[K: k(x_{r-1})]<2g$; on the other hand, N/N_1 is abelian and every element $\neq 1$ of this group is of order p.

By reducing to normal form the linear operation $\rho(s)$ on L for a given $s\in G(P)$, the authors construct certain rational subfields of K which are transformed into themselves by s. The existence of these subfields allows them to show that the order of s is finite and bounded (for all $s\in G(P)$). Considering the representation $s\to c(s)$, it follows immediately that $G(P)/N$ is cyclic of order m prime to p; it is then known that $m\leqq 6(2g-1)$. Moreover, it follows that N is a nilpotent group in which the order of every element is a power of p.

The authors then establish that an abelian group H of automorphisms of K in which the order of every element is a power of p is of finite order, bounded by $p^2(2g-1)$. This is proved by a reduction to the case where the fixed field of any non-trivial finite subgroup of H is rational, in which case H is established to be cyclic of order 1, p or p^2.

If the group N_1 introduced above is $\{1\}$, then the result quoted above proves the theorem. If not, then the fact that N is nilpotent shows that there exists a subgroup N_2 of index p in N_1, normal in N, such that N_1/N_2 is in the center of N/N_2. Because of the particular choice of x_{r-1}, it turns out that the fixed field K' of N_2 is of genus >0; the group N/N_2 is a group of automorphisms of K'. By a purely group-theoretic argument, it is established that N/N_2 contains comparatively big abelian subgroups (infinite if N is infinite,

of order at least $(pn)^{\frac{1}{2}}$ if N/N_2 is finite of order n). The theorem then follows by applying the result quoted above to an abelian subgroup of N/N_2.

Assume now that K is of genus >1, and let G be a group of automorphisms of K/k. Then G acts in a natural manner on the space D of differentials of the first kind of K. Let D_1 be a subspace of G which gives an irreducible representation δ of G; let G_1 be its kernel. If ω is $\neq 0$ in D_1, then the group G_2 of $s \in G$ such that $s\omega = a\omega$ (with $a \in k$) is finite, for the operations of this group permute among themselves the zeros of ω, from which it follows that a subgroup of finite index of G_2 is contained in some $G(P)$. It follows first that the order of any element of G is finite and secondly that G_1 is finite. The orders of the elements of G/G_1 being finite, it follows from a theorem of Burnside that G/G_1 itself is finite; G is therefore finite.

C. Chevalley (New York, N. Y.).

✓ ***Rademacher, Hans. Additive algebraic number theory.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 356–362. Amer. Math. Soc., Providence, R. I., 1952.

At the International Congress of Mathematicians, 1950, the author delivered an address on the development of analytic (additive) number theory for algebraic numbers, a branch which was hardly mentioned in his report on analytic number theory in 1941 [Bull. Amer. Math. Soc. **48**, 379–401 (1942); these Rev. **3**, 271]. This is an abstract of that address. The author begins by stating Waring's problem and Goldbach's problem for algebraic numbers. Comparing these with the case of rational integers, he explains how the classical Hardy-Littlewood method can be extended to algebraic number fields, using the equivalent of power series found by Hecke, and states contributions of Siegel and himself to these problems, noting that the formulae thus obtained show complete analogy to the results in rational cases. The author then discusses the problem of partition for algebraic numbers and shows that here the analogy to the rational case breaks down. For instance, the number $P(\nu)$ of partitions of an integer ν in a real quadratic field satisfies

$$P(\nu) = O(\exp N(\nu)^{2/3}),$$

which differs essentially from the asymptotic formula for the partition function $p(n)$ in the rational field. Showing the different characters of generating functions, the author explains the discrepancy between these two cases and notes that Hecke's "Grössencharaktere" will appear in the presumptive asymptotic formula for $P(\nu)$ in an algebraic number field.

K. Iwasawa (Princeton, N. J.).

Dirac, G. A. Note on a problem in additive number theory. J. London Math. Soc. **26**, 312–313 (1951).

Let $a_1 < a_2 < a_3 < \cdots$ be positive integers, $f(n)$ the number of representations of n as $a_i + a_j$, counting a representation twice if $i \neq j$, once if $i = j$. Erdös and Turán [same J. **16**, 212–215 (1941); these Rev. **3**, 270] have shown that $f(n)$ cannot be constant for $n \geqq n_0$. The author remarks that this follows directly from the parity of $f(2a_i)$ and of $f(2n+1)$. Let $\xi(n)$, $\eta(n)$ be the number of representations counting the case $i = j$ twice and not at all, respectively. It is proved that $\xi(n) = \xi(n_0)$ for all $n \geqq n_0$ is impossible and conjectured that $\eta(n) = \eta(n_0)$ for all $n \geqq n_0$ is also impossible.

N. J. Fine (Philadelphia, Pa.).

Davenport, H., and Erdös, P. On sequences of positive integers. J. Indian Math. Soc. (N.S.) **15**, 19–24 (1951).

Let $a_1, a_2, \cdots$ be an increasing sequence of positive integers, and let $A_m = A(a_1, \cdots, a_m)$ be the density of the sequence of integers which are divisible by any of $a_1, \cdots, a_m$. The authors have previously proved [Acta Arith. **2**, 147–151 (1936)] that $\lim_{m\to\infty} A_m$ exists and represents both the lower density and the logarithmic density of the sequence of integers divisible by any of $a_1, a_2, \cdots$. While the earlier proof made use of Tauberian theorem of Hardy and Littlewood, the authors now give an elementary proof, based on what they call the multiplicative density of a sequence $\{b_j\}$. This is defined as $\lim_{k\to\infty} (\sum 1/b_j')/(\sum 1/n')$, where b_j' runs over those elements of the sequence $\{b_j\}$ which are composed entirely of the first k primes, and n' runs over all the integers composed entirely of the first k primes.

W. J. LeVeque (Manchester).

Cheo, Luther. On the density of sets of Gaussian integers Amer. Math. Monthly **58**, 618–620 (1951).

Define θ as the set of Gaussian integers $u+iv$, $u \geqq 0$, $v \geqq 0$ but $u = v = 0$ excluded. Let A be a set of Gaussian integers in θ. Let $A(x+iy)$ denote the number of Gaussian integers $u+iv$ in A satisfying $u \leqq x$, $v \leqq y$. The density of A is defined as $\inf A(x+iy)/(xy+x+y)$ taken over all $x+yi$ in θ. Let A and B be two sets of Gaussian integers in θ, $A+B$ is defined as in the case of positive integers. Let α be the density of A and β the density of B. The author proves that if $\alpha+\beta \geqq 1$, then every $x+iy$, $x>0$, $y>0$, x, y integers, is in $A+B$. He also proves that if B contains all numbers ni, $n = 1, 2, \cdots$ then $\gamma \geqq \alpha_0 + \beta$, where γ is the density of $A+B$ and $\alpha_0 = \inf A(x+0i)/x$, or α_0 is the Schnirelmann density of the positive integers in A. The author further shows by an example that the $\alpha+\beta$ hypothesis does not hold in this case.

P. Erdös (Aberdeen).

Vijayaraghavan, T. On a problem in elementary number theory. J. Indian Math. Soc. (N.S.) **15**, 51–56 (1951).

Let $\phi(x, A)$ denote the number of positive integers which are $\leqq x$ and prime to A; $\phi(A) = \phi(A, A)$;

$$e_{A,n} = \phi(n, A) - n\phi(A)/A;$$

$\nu(A)$ denotes the number of different prime factors of A. The author shows that the elementary inequality $1 - 2^{\nu(A)-1} \leqq e_{A,n} \leqq 2^{\nu(A)-1}$ is best possible in the following sense. If $\delta > 0$ and the positive integer k are given, then positive integers n, m, A can be found such that $\nu(A) = k$, $\phi(A)/A > 1 - \delta$, $e_{A,n} < 1 - 2^{k-1} + \delta$, $e_{A,m} > 2^{k-1} - \delta$. This disproves a conjecture of Erdös [Math. Student **17**, 32–33 (1950); these Rev. **11**, 643].

N. G. de Bruijn (Delft).

de Bruijn, N. G. The asymptotic behaviour of a function occurring in the theory of primes. J. Indian Math. Soc. (N.S.) **15**, 25–32 (1951).

The function $\Psi(x, y)$, defined as the number of positive integers $\leqq x$ which are free of prime divisors $> y$, has been investigated by several writers in recent years. In particular, it was shown by S. D. Chowla and T. Vijayaraghavan [J. Indian Math. Soc. (N.S.) **11**, 31–37 (1947); these Rev. **9**, 332] that, for any fixed positive u,

$$\lim_{y\to\infty} y^{-u}\Psi(y^u, y) = \rho(u),$$

where $\rho(u)$ is continuous for $u>0$ and uniquely defined by the relations

(1) $\rho(u)=1 \quad (0\leqq u\leqq 1); \qquad u\rho'(u)=-\rho(u-1) \quad (u>1).$

In the present paper the author uses these relations to prove that, for $u\to\infty$,

$$(2)\qquad \rho(u)\sim\frac{e^{\gamma}}{\sqrt{(2\pi u)}}\exp\left\{-\int_0^{\xi}\frac{se^s-e^s+1}{s}ds\right\},$$

where γ is Euler's constant and, for $u>1$, ξ is the positive root of the equation $e^{\xi}-1=u\xi$. To obtain (2) we first note that, in virtue of (1),

$$u\rho(u)=\int_0^1\rho(u-t)dt \qquad (-\infty<u<\infty),$$

where we put $\rho(u)=0$ for $u<0$. Now an earlier paper due to the author [Nederl. Akad. Wetensch., Proc. 53, 813–821 (1950); these Rev. 12, 30] states that if $F(u)$, $F_0(u)$ are continuous and both satisfy

$$(3)\qquad uF(u)=\int_0^1 F(u-t)dt$$

for $u>0$ and if $F_0(u)$ is positive and non-increasing for $u\geqq 1$, then $F(u)=\{C+O(u^{-1})\}F_0(u)$ $(u\to\infty)$, where C is a constant. It therefore follows that

$$F_1(u)=\{C+O(u^{-1})\}\rho(u),$$

where $F_1(u)$ is any continuous solution of (3). Taking

$$F_1(u)=\frac{1}{2\pi i}\int_W\exp\left\{-uz+\int_0^z\frac{e^s-1}{s}ds\right\}dz,$$

where W is a suitable contour, and using the saddle point method we prove that

$$F_1(u)\sim\frac{1}{\sqrt{(2\pi u)}}\exp\left\{-\int_0^{\xi}\frac{se^s-e^s+1}{s}ds\right\}.$$

A second, and more difficult, application of the saddle point method leads to the relation $C=e^{-\gamma}$, and so to (2).

In addition to the asymptotic formula (2), the author also establishes an exact formula for $\rho(u)$, namely

$$\rho(u)=\frac{e^{\gamma}}{2\pi i}\int_{-i\infty}^{i\infty}\exp\left\{-uz+\int_0^z\frac{e^s-1}{s}ds\right\}dz \quad (u>0).$$

Here the outer integral is taken along the imaginary axis.

L. Mirsky (Bristol).

Bailey, W. N. On the simplification of some identities of the Rogers-Ramanujan type. Proc. London Math. Soc. (3) 1, 217–221 (1951).

In many of the identities discovered by Rogers, and recently by L. J. Slater [same Proc. (2) 53, 460–475 (1951); these Rev. 13, 227], the right side involves the sum of two infinite products. An example (due to Rogers) is

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)(1-q)(1-q^3)\cdots(1-q^{2n-1})}$$
$$=\prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)\left[\prod_{n=1}^{\infty}(1+q^{21n-10})(1+q^{21n-11})(1-q^{21n})\right.$$
$$\left.-q\prod_{n=1}^{\infty}(1+q^{21n-4})(1+q^{21n-17})(1-q^{21n})\right].$$

The author shows that in many cases the two products can be reduced to one by use of the formula

$$\prod_{n=1}^{\infty}((1+q^nz^{-1})(1+q^{n-1}z)(1-q^{2n-1}z^{-2})(1-q^{2n-1}z^2)(1-q^n)$$
$$=\prod_{n=1}^{\infty}(1-q^{2n-2}z^2)(1-q^{2n-1}z^{-2})(1-q^{2n})$$
$$+z\prod_{n=1}^{\infty}(1-q^{2n-2}z^{-2})(1-q^{2n-1}z^2)(1-q^{2n}),$$

which he proves. This formula is essentially the same as one given by Watson [J. London Math. Soc. 4, 39–48 (1929), pp. 44–45]. The author gives another, simpler formula which is frequently useful. *N. J. Fine.*

Gupta, Hansraj. A generalization of the partition function. Proc. Nat. Inst. Sci. India 17, 231–238 (1951).

Let $v_r(n)$ be defined by

$$nv_r(n)=\sum_{j=1}^{n}\sigma_r(j)v_r(n-j),$$

where $\sigma_r(j)$ is the sum of the rth powers of the divisors of j and $v_r(0)=1$. Thus $v_1(n)=p(n)$, the ordinary partition function. It is proved that

$$\sum_{n=0}^{\infty}v_r(n)x^n=\prod_{j=1}^{\infty}(1-x^j)^{-j^{r-1}}.$$

Thus $v_r(n)$ is the number of partitions of n when the part j is of j^{r-1} different types.

Similarly, if $v_r(n, m)$ is defined by

$$nv_r(n, m)=\sum_{j=1}^{n}\sigma_r(j, m)v_r(n-j, m),$$

where $\sigma_r(j, m)$ is the sum of the rth powers of those divisors of j which do not exceed m, then

$$\sum_{n=0}^{\infty}v_r(n, m)x^n=\prod_{j=1}^{m}(1-x^j)^{-j^{r-1}}.$$

Thus $v_r(n, m)$ is the number of partitions of n into parts not exceeding m, each part j being of j^{r-1} different types. If $m=o(n^{1/(2r+1)})$,

$$v_r(n, m)\prod_{j=1}^{m}j^{jr-1}\sim\binom{n-1}{S_{r-1}(m)-1},$$

where $S_k(m)=\sum_{i=1}^{m}i^k$. Also

$$v_2(n)=\exp\left[\{C+o(1)\}n^{2/3}\right],$$

where $C^3=(27/4)\zeta(3)$. There is a table of $v_2(n, m)$ for $m\leqq n\leqq 50$. *N. J. Fine* (Philadelphia, Pa.).

Fawaz, A. Y. The explicit formula for $L_0(x)$. Proc. London Math. Soc. (3) 1, 86–103 (1951).

Let

$$L(x)=\sum_{n\leqq x}\lambda(n), \quad L_0(x)=\tfrac{1}{2}[L(x+0)+L(x-0)]=\sum_{n\leqq x}{}'\lambda(n),$$

where $\lambda(n)$ is Liouville's arithmetical function defined by

$$\frac{\zeta(2s)}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s} \quad (s=\sigma+it,\ \sigma>1)$$

and $\sum'$ indicates that, when x is an integer, the term corresponding to $n=x$ is to have the factor $\frac{1}{2}$. The functions $L(x)$ and $L_0(x)$ differ only when x is an integer, their difference then being $\pm\frac{1}{2}$. It has been conjectured by Pólya [Jber.

Deutsch. Math. Verein. **28**, 31–40 (1919)] that $L(x) \leqq 0$ for $x \geqq 2$, and by Ingham [Amer. J. Math. **64**, 313–319 (1942); these Rev. **3**, 271] that lim sup (inf) $L(x)/x^{\frac{1}{2}} = +\infty\,(-\infty)$. In the present paper the author gives an explicit formula for $L_0(x)$ on the assumption that the Riemann hypothesis is true and that the zeros of $\zeta(s)$ are all simple.

W. Simons (Vancouver, B. C.).

Apostol, T. M. Remark on the Hurwitz zeta function. Proc. Amer. Math. Soc. **2**, 690–693 (1951).

The author uses the functional equation for the function $\phi(x, a, s) = \sum_{n=0}^{\infty} e^{2\pi n i x}(a+n)^{-s}$ [Lerch, Acta Math. **11**, 19–24 (1887)] in order to prove the formula

$$\zeta(1-s, a) = 2\Gamma(s)(2\pi)^{-s}\sum_{1}^{\infty} n^{-s}\cos\left(\tfrac{1}{2}\pi s - 2\pi a n\right) \qquad (0 < a \leqq 1,\ \Re s > 1)$$

[see Whittaker and Watson, A course of modern analysis . . . , Macmillan, New York, 1943, p. 269]. Here $\zeta(s, a)$ represents the Hurwitz zeta function, defined by $\zeta(s, a) = \sum_{0}^{\infty}(a+n)^{-s}$ $(0 < a \leqq 1, \Re s > 1)$.

N. G. de Bruijn (Delft).

Apostol, T. M. On the Lerch $z_{\epsilon+\zeta}$ function. Pacific J. Math. **1**, 161–167 (1951).

The author gives a new proof of Lerch's functional equation [see Acta Math. **11**, 19–24 (1887)] for the function $\phi(x, a, s)$, defined by analytic continuation of the series $\sum_{0}^{\infty} e^{2\pi n i x}(a+n)^{-s}$. His proof depends on the use of the transformation formula for $\vartheta_3(y|\tau)$ and of the differential-difference equations satisfied by ϕ. The values of ϕ for $s = 0, -1, -2, \cdots$ are studied in detail. If

$$\beta_n(a, \alpha) = -n\phi(x, a, 1-n) \quad (\alpha = e^{2\pi i x};\ n = 1, 2, \cdots),$$

then $(\alpha-1)^n\beta_n(a, \alpha)$ turns out to be a polynomial in a and α. A number of properties of $\beta_n(a, \alpha)$ are derived, most of which are generalizations of properties of the Bernoulli numbers $B_n(a) = \beta_n(0, a)$.

N. G. de Bruijn (Delft).

Emersleben Otto. Über die Konvergenz der Reihen Epsteinscher Zetafunktionen. Math. Nachr. **4**, 468–480 (1951).

The author discusses the convergence of the Epstein zeta function

$$\sum_{k_1, \cdots, k_p = -\infty}^{\infty}{}' \frac{e^{2\pi i(h_1 k_1 + \cdots + h_p k_p)}}{(k_1^2 + \cdots + k_p^2)^{\alpha/2}}$$

where the numbers h_r are real and the prime indicates that all the summation indices cannot simultaneously be zero. The author quotes known results of Walfisz, Landau and Hua on exponential sums and the number of lattice points on a p-dimensional sphere to give trivial proofs of a number of results of which the following are typical. Let α be the abscissa of convergence of the above Dirichlet series and let β be the abscissa of absolute convergence. (i) If all h_r are rational and not all are integral, then $\beta - 2 \leqq \alpha \leqq p-2$ if $p \geqq 4$. (ii) If all h_r are arbitrary real numbers then $\beta - 2 \leqq \alpha \leqq p - 2 + 2/(p+1)$ if $p \geqq 2$.

L. Schoenfeld.

van der Blij, F. The function $\tau(n)$ of S. Ramanujan (an expository lecture). Math. Student **18**, 83–99 (1950).

"Somewhat enriched translation" of Math. Centrum Amsterdam. Rapport **ZW1948-010** (1948); these Rev. **10**, 514.

Vinogradov, I. M. General theorems on the upper bound of the modulus of a trigonometric sum. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 109–130 (1951). (Russian)

This paper is devoted to the determination of upper bounds for sums and integrals of the form

$$S = \sum_{x=1}^{P} \exp\left[2\pi i\{\alpha_{n+1}x^{n+1} + f(x)\}m\right],$$

$$H = \int_0^1 \cdots \int_0^1 |S|^{\tau} d\alpha_n \cdots d\alpha_1$$

when $f(x) = \alpha_n x^n + \cdots + \alpha_1 x$ and $m = 1$. Particular attention is paid to the determination of the parameters involved in the O-estimates for S and H. Such explicit determinations of the parameters are needed, for example, in the estimation of $\zeta(s)$ near the line $\sigma = 1$ and a number of such results have been given by Hua [Quart. J. Math., Oxford Ser. (1) **20**, 48–61 (1949); these Rev. **10**, 597] and Flett [Quart. J. Math., Oxford Ser. (2) **2**, 26–52 (1951); these Rev. **13**, 209].

Lemmas 3 and 7 of this paper estimate H explicitly, thereby providing mean-value theorems which may be compared with a related O-estimate on p. 73 of Vinogradov's book [Trudy Mat. Inst. Steklov **23** (1947); these Rev. **10**, 599], with the O-estimate on p. 49 of Hua's book [Trudy Mat. Inst. Steklov **22** (1947); these Rev. **10**, 597] and the explicit estimate of Theorem 1 of Hua's paper.

Theorem 1a explicitly estimates S when $\alpha_{n+1} = 0$, $m = 1$, and suitable upper and lower bounds for $|f^{(n+1)}(x)|$ are known; this result is to be compared with the O-estimate on p. 68 of Vinogradov's book and with Flett's Theorem B which has a wider range of applicability. Theorems 2–5 deal with the explicit estimation of S when $f(x) = \alpha_n x^n + \cdots + \alpha_1 x$ and diophantine inequalities are known for the real coefficients α_k; the number m is generally unrestricted (because of possible application to questions of uniform distribution and related problems). More specifically, Theorems 2 and 4 treat the case in which for some s between 2 and $n+1$ it is known that $|\alpha_s - a/q| \leqq 1/q^2$ with $(a, q) = 1$ and $1 < q < P^s$; the results are similar to the O-estimate on p. 63 of Vinogradov's book and may be compared with the O-estimates on pp. 65 and 68 of Hua's book. A novel feature is introduced in Theorems 3 and 5 in which it is assumed that $|\alpha_s - a_s/q_s| \leqq 1/(Pq_s)$ with $(a_s, q_s) = 1$, $0 < q_s \leqq P$ for each $s = 2, \cdots, n+1$; the explicit estimate is now given in terms of Q, the least common multiple of $q_2, \cdots, q_{n+1}$ provided $m \leqq P^{0.5\tau/n}$ where $\tau = 1$ if $Q \geqq P$ and $\tau = \log Q/\log P$ otherwise. Under the above assumptions, Theorem 3 gives the result $|S| \leqq (8n)^{8nl} P^{1-\rho}$ where $\rho = \tau\{3n(n+1)l\}^{-1}$ and $l = \log\{12(n+1)^2/\tau\}$. The proofs depend in some measure on a previous paper of the author [Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 199–214 (1950); these Rev. **12**, 161] and follow methods used previously by the author.

L. Schoenfeld (Urbana, Ill.).

Hua, L. K., and Reiner, I. Automorphisms of the unimodular group. Trans. Amer. Math. Soc. **71**, 331–348 (1951).

Let $\mathfrak{M}_n$ denote the unimodular group of order n (elements are the $n \times n$ matrices of determinant ± 1) and let $\mathfrak{A}_n$ be the group of its automorphisms. Let $\mathfrak{M}_n^+$ and $\mathfrak{M}_n^-$ stand for the subsets of $\mathfrak{M}_n$ with determinants $+1$ and -1, respectively; let $\mathfrak{R}_n$ be the commutator subgroup of $\mathfrak{M}_n$, $I^{(n)}$ the identity matrix of order n and X' the transpose of X. The authors determine the generators of $\mathfrak{A}_n$ explicitly, by proving: (1) For $n > 2$, $\mathfrak{R}_n = \mathfrak{M}_n^+$; for $n = 2$, $\mathfrak{R}_n$ is of index 2 in

$\mathfrak{M}_n^+$. Corollary: Under any automorphism of $\mathfrak{M}_n$, $\mathfrak{M}_n^+ \to \mathfrak{M}_n^+$. (2) The automorphism group of $\mathfrak{M}_2^+$ is generated by the set of "inner" automorphisms $X \to AXA^{-1}$ ($A \in \mathfrak{M}_2$) and $X \to \epsilon(X) \cdot X$ ($\epsilon(X) = \pm 1$). (3) For $n > 2$ the group of automorphisms of $\mathfrak{M}_n^+$ induced by automorphisms of $\mathfrak{M}_n$ is generated by the set of "inner" automorphisms $X \to AXA^{-1}$ ($A \in \mathfrak{M}_n$) and the automorphism $X \to X'^{-1}$. (4) The generators of $\mathfrak{A}_n$ are (i) the inner automorphisms $X \to AXA^{-1}$ ($A \in \mathfrak{M}_n$); (ii) $X \to X'^{-1}$; (iii) for even n only, $X \to (\det X) \cdot X$; (iv) for $n = 2$ only, the automorphism $X \to \epsilon(X) \cdot X$, if $X \in \mathfrak{M}_2^+$, or $X \to \epsilon(JX) \cdot X$, if $X \in \mathfrak{M}_2^-$, with $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The proof of (1) is comparatively simple and uses a previous result of the same authors [Trans. Amer. Math. Soc. 65, 415–426 (1949); these Rev. 10, 684]. (2) follows by establishing it first directly for the two generators of $\mathfrak{M}_2^+$. (4) follows easily from (2), (3) and the corollary. The real difficulty lies in the proof of (3). This consists of two parts. First, a direct proof for the case $n = 3$ is given; then the proof is completed by induction on n, using two lemmas of some independent interest.

E. Grosswald (Princeton, N. J.).

Obrechkoff, N. Sur l'approximation des nombres irrationnels par des nombres rationnels. C. R. Acad. Bulgare Sci. 3, no. 1 (1950), 1–4 (1951). (Russian. French summary)

Let

$$\omega = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}$$

be an irrational number, and let p_n/q_n be the nth convergent (where $p_1 = a_1$, $q_1 = 1$). The author proves very simply that, for any n, the inequality

$$|\omega - p_n/q_n| < q_n^{-2}(4 + a_n^2)^{-1}$$

holds for one at least of the three values $m = n-2, n-1, n$. (For an account of known results, some of which are very similar to this one, see J. F. Koksma, Diophantische Approximationen [Ergebnisse der Mathematik und ihrer Grenzgebiete, v. 4, Springer, Berlin, 1936], chapter 3.)

H. Davenport (London).

ANALYSIS

Kaluza, Th., jr. Ein allgemeiner Satz über die Existenz von Mittelwertfunktionen. Arch. Math. 2 (1949–1950), 438–440 (1951).

It is shown that to any real function $f(x)$ defined on an interval (a, b) there necessarily corresponds at least one mean-value function, that is, a function $\varphi(x)$ such that to each x_1, x_2 with $a < x_1 < x_2 < b$ there corresponds a ξ, $x_1 < \xi < x_2$, for which $(x_2 - x_1)\varphi(\xi) = f(x_2) - f(x_1)$. The function $\varphi(x)$ can further be restricted to assuming all values between $\varphi(x_1)$ and $\varphi(x_2)$ for $x_1 < x < x_2$.

E. F. Beckenbach (Princeton, N. J.).

Karamata, Jovan. Sur la formule des accroissements finis. Srpska Akad. Nauka. Zbornik Radova, Knj. 7. Matematički Institut, Knj. 1, 119–124 (1951). (Serbo-Croatian. French summary)

The author gives a proof of the following extension of the law of the mean: If $f(x)$ is continuous in the closed interval (a, b) and admits a left derivative $f'_-(x)$ and a right derivative $f'_+(x)$ at each point of the open interval, then there exist values p, q, ξ with $p > 0$, $q > 0$, $p + q = 1$, $a < \xi < b$, such that

$$\frac{f(b) - f(a)}{b - a} = pf_+'(\xi) + qf_-'(\xi)$$

E. F. Beckenbach (Los Angeles, Calif.).

Obrechkoff, Nikola. Sur quelques propriétés des fonctions réelles définies sur tout l'axe réel. C. R. Acad. Bulgare Sci. 3, no. 1 (1950), 5–8 (1951). (Russian. French summary)

The author proves the following theorems. (I) Let $f(x)$ be a real-valued function on $(-\infty, \infty)$ with a non-negative nth derivative. Let $\{y_k\}_{-\infty}^{\infty}$ be a sequence of points tending to $\pm\infty$ as $k \to \pm\infty$, and suppose that for some m, $0 \leqq m < n$, $\lim_{k \to \pm\infty} f(y_k) y_k^{-m} = 0$. Then $f(x)$ is a polynomial of degree at most $m-1$. (II) If n is even and $f(y_k) < K x_k^{n-2}$, then $f(x)$ is a polynomial of degree at most $n-1$. Several corollaries are given, for example that if $f(x)$ is twice differentiable, if $f(y_k) < K \exp(-y_k^2)$ and if $f''(x) + 4f'(x) + 2(1 + 2x^2)f(x) \geqq 0$ then $f(x) = Ce^{-x^2}$. Analogous theorems involving sequences and their differences are indicated.

R. P. Boas, Jr.

Korenblyum, B. I. On two theorems from the theory of absolutely monotonic functions. Uspehi Matem. Nauk (N.S.) 6, no. 4 (44), 172–175 (1951).

[What the author calls an absolutely monotonic function is usually called a completely monotonic function.] This note contains simplified proofs of the theorems of S. Bernstein and Widder that $f(x) = \int_0^\infty e^{-xt} d\alpha(t)$, $0 \leqq x < \infty$, with $\alpha(t)$ respectively nondecreasing and bounded or of bounded variation on $0 \leqq t < \infty$, if (and only if), respectively, $(-1)^n f^{(n)}(x) \geqq 0$, $0 \leqq x < \infty$, or

$$\int_0^\infty x^n |f^{(n+1)}(x)|\, dx \leqq Mn!, \quad n = 0, 1, 2, \cdots.$$

The simplification results from two remarks: first, that $\phi_n(x) = (1 - x/n)^n$, $0 \leqq x < n$, $\phi_n(x) = 0$ elsewhere, tends uniformly to e^{-x}; second, that the integral formula for expressing a function in terms of its nth derivative leads to

$$f(x) - f(\infty) = \int_0^\infty \phi_n(xt)\, d\alpha_n(t),$$

where $\alpha_n(t)$ is essentially the integral of the Post-Widder inversion operator $L_{n+1,t}[f(x)]$ [Widder, The Laplace transform, Princeton University Press, 1941, p. 288; these Rev. 3, 232]. The result now follows in the usual way by using Helly's theorem on families of functions of uniformly bounded variation.

R. P. Boas, Jr. (Evanston, Ill.).

Tagamlickiĭ, Ya. A. On a generalization of Abel's series. Doklady Akad. Nauk SSSR (N.S.) 80, 17–20 (1951). (Russian)

Let $\{x_n\}$ be an arithmetic progression with difference τ, $x_0 > a$; the author calls $f(x)$ positive definite if $(-1)^k f^{(k)}(x) \geqq 0$ for $a < x \leqq x_k$, $k = 0, 1, 2, \cdots$; this is an extension of the notion of a completely monotonic function. A partial ordering is introduced by putting $f_2(x) \subset f_1(x)$ if $f_1(x) - f_2(x)$ is positive definite, and $f(x)$ is called indecomposable if $0 \subset \phi(x) \subset f(x)$ implies $\phi(x) = Cf(x)$ (constant C). The author has previously shown [same Doklady (N.S.) 75, 337–340 (1950); these Rev. 12, 396] that besides the Abel Polynomials $P_n(x) = (x_0 - x)(x_n - x)^{n-1}/n!$ the functions

$$(x_0 - x) \exp[(x_0 - x)/\tau] = \lim n! P_n(x) n^{-n+1} \tau^{-n+1}$$

are indecomposable; he states here that in addition the functions $R(x,t)=\{\exp[\lambda(x_0-x)]-\exp[\mu(x_0-x)]\}/(\lambda-\mu)$ with $\tau\lambda e^{1-\lambda\tau}=\tau\mu e^{1-\mu\tau}=t$, $0<\mu<\tau^{-1}<\lambda$, are indecomposable, and that these three classes (and their constant multiples) exhaust the class of indecomposable functions. His main result is an expression of every positive definite function in terms of indecomposable functions: $F(x)$ is positive definite if and only if

$$F(x)=\sum_{n=0}^{\infty}a_nP_n(x)+\int_{0+}^{1}R(x,t)d\alpha(t),$$

where $R(x,1)$ is to be interpreted as $(x_0-x)\exp[(x_0-x)/\tau]$, a_n are nonnegative and $\alpha(t)$ is nondecreasing.

R. P. Boas, Jr. (Evanston, Ill.).

Dugué, Daniel. Démonstration par la théorie des familles normales d'un théorème de M. S. Bernstein et de résultats analogues. Bull. Sci. Math. (2) **75**, 153–160 (1951).

The theorem in question is that if $f(x)$ has all its derivatives positive for $0\leqq x\leqq 1$ then it is analytic and its Maclaurin series has radius of convergence at least 1. The author uses the normal family formed by the partial sums of the Maclaurin series. The main part of the proof consists of showing that the Maclaurin series represents the right function. The author then proves two theorems on the analyticity of the limit of a sequence of polynomials with positive coefficients, stated in an earlier note [C. R. Acad. Sci. Paris **228**, 1469–1470 (1949); these Rev. **10**, 516]. He proves three theorems on the analyticity of functions with a sequence of positive derivatives, of which one is as follows. If $f^{(n)}(x)\geqq 0$, $0\leqq x\leqq 1$, for $n=n_k\uparrow\infty$, and $f^{(n)}(0)=\phi^{(n)}(0)$ for all n, where $\phi(x)$ is analytic in $|x|<1$, then $f(x)$ is analytic. His corollary [p. 160] and conjecture [p. 156] are special cases of the theorem of Bernstein that $f(x)$ is analytic if $f^{(n)}(x)\geqq 0$ for $n=n_k$ where n_{k+1}/n_k is bounded [Leçons sur les propriétés extrémales . . . , Gauthier-Villars, Paris, 1926, p. 197; cf. Boas, Duke Math. J. **8**, 163–172 (1941); these Rev. **2**, 351]. *R. P. Boas, Jr.*

MacDuffee, C. C. Families of Lorentzian matrices. Proc. Amer. Math. Soc. **2**, 794–797 (1951).

Let J be a (n by n) real, symmetric, orthogonal matrix. Let $P(s)$, for $a\leqq s\leqq b$, be a continuous real matrix such that $P(s)J$ is skew. Let $A=A(s)$ satisfy the matrix differential equation (1) $A'=PA$ and let A^T be the transpose of A. Then (2) $A^TJA=J$ holds for $a\leqq s\leqq b$ if it holds for one value of s. Conversely, if $A(s)$ is a continuously differentiable matrix for $a\leqq s\leqq b$ satisfying the identity (2), then there exists a unique continuous matrix $P(s)$ with the properties that PJ is skew and (1) holds. [For the case when J is the unit matrix, cf. Wintner, The Analytical Foundations of Celestial Mechanics, Princeton Univ. Press, 1941, pp. 50–52; these Rev. **3**, 215. The direct result, for this case, is a standard one which occurs in connection with the Frenet equations of a curve.] *P. Hartman* (Baltimore, Md.).

Theory of Sets, Theory of Functions of Real Variables

Shepherdson, J. C. Well-ordered sub-series of general series. Proc. London Math. Soc. (3) **1**, 291–307 (1951).

The "index", $\iota(S)$, of an ordered set S is defined to be the least ordinal number greater than the ordinals of all well-ordered subsets of S. If $\alpha>0$, there exists an S with $\iota(S)=\alpha$; and if $\omega\leqq\alpha\leqq\omega_{\beta+1}$, there exists an S with $|S|=\aleph_\beta$ and $\iota(S)=\alpha$. If $1\leqq\alpha<\omega$, then $\iota(S)=\alpha$ if, and only if, $|S|=\alpha-1$; $\iota(S)=\omega$ if, and only if, S^* (obtained from S by reversing the order) is a well-ordered set whose ordinal is not less than ω. Let $S(\iota)$ mean an ordered set whose index is ι. Suppose that the ordinal α has the normal form $\sum_{i=1}^{k}\omega^{\beta_i}n_i$. Then $\iota(S)=\alpha$ if, and only if,

$$S=\sum_{i=1}^{k-1}\left(\sum_{j=1}^{n_i}S_{ij}(\omega^{\beta_i}+1)\right)+\sum_{j=1}^{n_k-1}S_{kj}(\omega^{\beta_k}+1)+S_0(\omega^{\beta_k}),$$

and this expression for S is unique. This shows that if $\iota(S)$ is not of the form ω^β or $\omega^\beta+1$, then S can be expressed as a finite sum of ordered sets with indices smaller than $\iota(S)$. Necessary and sufficient conditions are obtained for $\iota(S)$ to equal $\omega^\beta+1$ ($\beta>0$), which show that in this case S can be expressed as a sum of ordered sets with indices smaller than $\iota(S)$ over a well-ordered set with ordinal number less than $\iota(S)$. If $\iota(S)=\omega^\beta$ and ω^β is not regular, S can be decomposed into a sum of ordered sets with indices less than $\iota(S)$ over an ordered set with index less than $\iota(S)$; if ω^β is regular, however, a decomposition of this kind can sometimes be obtained, sometimes not, and results are given which restrict the possible form of an ordered set with such an index.

F. Bagemihl (Rochester, N. Y.).

Sierpiński, Waclaw. Une proposition de la géométrie élémentaire équivalente à l'hypothèse du continu. C. R. Acad. Sci. Paris **232**, 1046–1047 (1951).

The author proves that the continuum hypothesis is equivalent to the following statement: The unit cube is the union of three sets E_1, E_2, E_3 where E_1 is intersected in a finite set by every line parallel to the X-axis, E_2 is intersected in a finite set by every line parallel to the Y-axis and E_3 is intersected in a finite set by every line parallel to the Z-axis. *P. Erdős* (Aberdeen).

Sierpiński, W. Sur une propriété des ensembles plans équivalente à l'hypothèse du continu. Bull. Soc. Roy. Sci. Liège **20**, 297–299 (1951).

The author proves that the hypothesis of the continuum is equivalent to the following statement: Every planar set E of power less than c is the union of two sets A and B, where A is intersected in a finite set by every line parallel to the X-axis and B is intersected in a finite set by every line parallel to the Y-axis. *P. Erdős* (Aberdeen).

Novák, J. A paradoxical theorem. Fund. Math. **37**, 77–83 (1950).

The author proves a theorem concerning ordinal numbers [which is actually a generalization of a theorem due to Alexandroff and Urysohn; see the following review], from which he derives the following theorem referred to in the title. Let S be a nonempty set, and define sets A_α, B_α, by induction on α, as follows. Let A_0 be a nonempty, at most enumerable subset of S; $B_0=0$. Let $\beta>0$, and suppose that A_α, B_α have already been defined for every $\alpha<\beta$. If the set $\sum_{\alpha<\beta}A_\alpha-\sum_{\alpha<\beta}B_\alpha$ is not empty, let B_β be a nonempty subset of it, and let A_β be an at most enumerable subset of $S-\sum_{\alpha<\beta}A_\alpha$. Then there exists an ordinal ξ, $1<\xi<\omega_1$, such that $\sum_{\alpha<\xi}A_\alpha=\sum_{\alpha<\xi}B_\alpha$. The above-mentioned result on ordinals is also used to obtain two necessary and sufficient conditions for a continuous ordered set possessing the Souslin property of having no nonenumerable class of non-overlapping intervals to be the continuum.

F. Bagemihl (Rochester, N. Y.).

Neumer, Walter. Verallgemeinerung eines Satzes von Alexandroff und Urysohn. Math. Z. **54**, 254–261 (1951).

Let ω_α be an initial number greater than ω, $M \subseteq W(\omega_\alpha)$, $M^* = W(\omega_\alpha) - M$. If M is similar to $W(\omega_\alpha)$, M is called a "full part" (of $W(\omega_\alpha)$). M is said to be "closed" (in $W(\omega_\alpha)$), if M contains the limit of every fundamental sequence in M whose limit is less than ω_α. A closed, full part of $W(\omega_\alpha)$ is termed a "band" (of $W(\omega_\alpha)$). Now assume that 0 non-ϵM, and to every $\xi \epsilon M$ let there correspond an ordinal number $\varphi(\xi) < \xi$. Dushnik [Bull. Amer. Math. Soc. **37**, 860–862 (1931)] proved that if α is of the form $\beta+1$, and $M^* = \{0\}$, then there is a value of φ which is assumed for $\aleph_\alpha$ ordinals in M. (The case $\alpha = 1$ is the theorem referred to in the title.) The author proves the following: (1) Let ω_α be regular, and M be a full part. If M^* contains no band, then there is a value of φ which is assumed $\aleph_\alpha$ times in M; but if M^* contains a band, a function $\varphi(\xi) < \xi$ can be defined in M which assumes no value $\aleph_\alpha$ times. (2) If ω_α is singular, then in every full part M a function $\varphi(\xi) < \xi$ can be defined which assumes no value $\aleph_\alpha$ times. (3) If $\omega_\alpha > \omega_1$, there exists a full part M such that M^* is also a full part and neither M nor M^* contains a band. (4) If ω_α is regular and M is a full part which contains a band, then M^* contains no band (and in this special case, the first half of (1) can be proved by Dushnik's method; for this part of (1) for $\alpha = 1$, see the paper by Novák reviewed above. *F. Bagemihl.*

Kapuano, Isaac. Sur un problème de M. Sierpiński. C. R. Acad. Sci. Paris **232**, 1621–1622 (1951).

Answering two problems of Sierpiński, the author shows the following. (1) Given an unbounded linear set E and a sequence of linear sets, there is at least one set in the sequence which is not a translation of E. (2) Using translations and rotations a similar statement holds for plane sets. *J. F. Randolph* (Rochester, N. Y.).

***Izumi, Shin-ichi. Jitsu-kansûron. [Theory of Real Functions].** Jibundo, Tokyo, 1947. iii+273 pp. 160 yen.

I. Theory of sets. II. Theory of measure. III. Functions of a real variable. IV. Theory of the integral. V. Fourier analysis. Appendix: General set theory.

Table of contents.

Nikodým, Otton Martin. Remarks on the Lebesgue's measure extension device for finitely additive Boolean lattices. Proc. Nat. Acad. Sci. U. S. A. **37**, 533–537 (1951).

Cette note réunit et complète les principaux résultats obtenus depuis 1938 en vue d'étendre la théorie de la mesure extérieure au cas d'une algèbre de Boole, ou "tribu de Boole". L'auteur part d'une mesure simplement additive μ définie dans une tribu (A) pour parvenir à une mesure extérieure μ_e définie dans une tribu (A') prolongeant (A), par le processus classique des recouvrements d'un élément de (A') au moyen d'une infinité d'éléments de (A). Supposant d'abord (A') dénombrablement additive, il définit les ensembles mesurables par la méthode de Lebesgue, par celle de Carathéodory et par celle qu'il a lui-même introduite [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. **17**, no. 7 (1938)]. Ces ensembles forment dans les trois cas une même tribu (L') sur laquelle μ_e se réduit à une mesure μ^L dénombrablement additive. L'auteur indique quatre conditions équivalentes classiques pour que μ^L constitue un prolongement de μ.

Il suppose ensuite (A') simplement additive: s'appuyant sur un résultat de MacNeille [Proc. Nat. Acad. Sci. U. S. A. **24**, 188–193 (1938)], il définit une mesure extérieure particulière dite "faible" ayant des propriétés qu'il parait difficile de résumer ici. Les démonstrations ne sont qu'esquissées. *R. de Possel* (Alger).

Papoulis, A. On the density theorem. Proc. Amer. Math. Soc. **2**, 709–717 (1951).

Associons à chaque point P du plan une suite I_n de rectangles contenant P et de diamètres tendant vers zéro. La densité d'un ensemble mesurable F évalueé avec ce système est-elle en presque tout point de F égale à 1? On sait (a) qu'il en est ainsi si les côtés des I_n ont des directions fixes, mais (b) qu'il n'en est plus de même si les côtés des I_n sont seulement assujettis à faire un angle inférieur à $\theta > 0$ avec deux directions fixes rectangulaires [voir, par exemple, Busemann et Feller, Fund. Math. **22**, 226–256 (1934)]. L'auteur démontre (c) que ce dernier résultat subsiste si la direction des côtés des rectangles est fixe pour chaque point mais peut varier d'un point à un autre. Il construit d'abord un ensemble F et un système de rectangles démontrant à nouveau (b), puis, par des constructions analogues, il construit un exemple démontrant (c). *R. de Possel.*

Dvoretzky, A., Wald, A., and Wolfowitz, J. Relations among certain ranges of vector measures. Pacific J. Math. **1**, 59–74 (1951).

Let X be a set with a prescribed Boolean σ-algebra of sets (called measurable); let n and p be positive integers. If $f_1, \cdots, f_n$ are non-negative measurable functions on X such that $f_1 + \cdots + f_n = 1$, and if $\mu_1, \cdots, \mu_p$ are measures in X, write v for the np-vector with coordinates $\int f_i d\mu_j$. For fixed $\mu_1, \cdots, \mu_p$, the authors denote by V, V^0, and V^* the range of v for all systems of functions f, for all systems of simple functions, and for all systems of characteristic functions, respectively. If the measures μ are finite, then (Theorem 1) V is compact and convex, (Theorem 2) every extreme point of V belongs to V^*, and (Theorem 3) $V = V^0$. If, moreover, the measures μ are non-atomic, then (Theorem 4) $V = V^*$. The last conclusion is valid also in the infinite case (Theorem 5). *P. R. Halmos* (Montevideo).

Bellman, Richard, and Blackwell, David. On moment spaces. Ann. of Math. (2) **54**, 272–274 (1951).

The authors present an explicit description of two sets X and Y of vectors; X consists of all vectors of the form $(x_0, \cdots, x_n)$ where $x_i = \int g^i dt$ and E is a Borel subset of the unit interval, and Y consists of all vectors of the form $(y_0, \cdots, y_n)$ where $y_i = \int_E g^i dt$ and $\{E_0, \cdots, E_n\}$ is a partition of the unit interval into Borel sets. *P. R. Halmos.*

Ellis, H. W. On the compatibility of the approximate Perron and the Cesàro-Perron integrals. Proc. Amer. Math. Soc. **2**, 396–397 (1951).

An example is given of two functions $F(x)$ and $G(x)$ such that $F(x)$ $(G(x))$ is approximately continuous (Cesàro continuous) everywhere, $F(x) = G(x)$ if $x > 0$, $F(x) = 0$ and $G(x) = 1$ if $x \leq 0$, and the derivatives $DF(x) = DG(x)$ exist if $x \neq 0$. *M. Cotlar* (Chicago, Ill.).

Schärf, H. M. On the equivalence of two types of Stieltjes integrals. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 220–222 (1951).

Soit f une fonction réelle définie dans l'intervalle fermé J, D une division de J par des points x_i, et

$$S_D = \sum f(x_i)[g(x_{i+1}) - g(x_i)].$$

On dit que f a une intégrale à gauche de Cauchy-Stieltjes L du type normal si L est la limite des sommes S_D quand le plus grand des intervalles de D tend vers zéro, et du type "refinement" si, pour tout $\epsilon>0$, il existe une division D_ϵ dont toute subdivision D vérifie $|L-S_D|<\epsilon$. L'auteur démontre le théorème d'équivalence suivant: si f est bornée et g à variation bornée, l'existence du type "refinement" par rapport à g équivaut à l'existence du type normal par rapport à g_r, fonction de continuité à droite de g. Il en déduit une condition nécessaire et suffisante pour l'existence du type "refinement" donnée par Deniston [Indagationes Math. 11, 385–393, 394–402 (1949); ces Rev. 11, 587].

R. de Possel (Alger).

Ridder, J. Bemerkungen zur vorangehenden Note von H. Schärf. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 223–225 (1951).

L'auteur insiste de nouveau sur l'intérêt des ϵ-partages qu'il a introduits précédemment dans la définition de l'intégrale de Riemann-Stieltjes. Il en rappelle les nombreux avantages [Nederl. Akad. Wetensch., Proc. 52, 1129–1134 (1949); ces Rev. 11, 587]. Il montre que les résultats de Deniston [ibid. 52, 1111–1119, 1120–1128 (1949); ces Rev. 11, 587] ainsi que le théorème d'équivalence de Schärf de la note analysée ci-dessus s'obtiennent aisément par la méthode des ϵ-partages. *R. de Possel* (Alger).

de Bruijn, N. G. Functions whose differences belong to a given class. Nieuw Arch. Wiskunde (2) 23, 194–218 (1951).

The author considers real functions $f(x)$, defined for $-\infty<x<\infty$, with the property that $f(x+h)-f(x)$ belongs for each h to a given class C, and asks when this fact implies $f(x)=g(x)+H(x)$, where $g(x)\in C$ and $H(x)$ is additive, i.e. $H(x+y)=H(x)+H(y)$. He shows that, among others, the classes of continuous functions, k times differentiable functions, analytic functions, absolutely continuous functions, and functions of bounded variation have the property in question. On the other hand the classes of measurable functions or of bounded functions do not have the property. The class of functions belonging to L^2 over every finite interval has the weaker property that whenever $f(x+h)-f(x)$ belongs to the class for each h, we have $f(x)=g(x)+H(x)+S(x)$, where $g(x)$ is in the class, $H(x)$ is additive, and for each h the relation $S(x+h)=S(x)$ holds for almost all x. Since an additive function which is bounded on a set of positive measure is linear [cf. Kestelman, Fund. Math. 34, 144–147 (1947); these Rev. 9, 188], the positive theorems show in particular that the equation $f(x+h)-f(x)=$function belonging to C for each h has only solutions belonging to C if $f(x)$ is bounded on a set of positive measure. Many other results, both special and general, are given.

R. P. Boas, Jr. (Evanston, Ill.).

Popoviciu, Tiberiu. Sur les fonctions d'une variable réelle dont l'ensemble de définition est la réunion de deux sous-ensembles de monotonie opposée. An. Acad. Repub. Pop. Române. Secţ. Şti. Mat. Fiz. Chim. Ser. A. 3, 1–16 (1950). (Romanian. Russian and French summaries)

The author first gives a new proof of his known result [Acad. Roum. Bull. Sect. Sci. 20, 45–49 (1939); these Rev. 1, 71] that the set E of definition of a real function of a real variable is decomposable into two consecutive subsets, on which the function is monotone in opposite senses, if and only if this property holds for each set of three points of E. Next, dropping the requirement that the subsets be consecutive, he points out that there are functions defined on sets E of four points for which the required decomposition is not possible; but he shows that if E contains more than four points then the decomposition is possible for E if and only if it is possible for every set of five points of E.

E. F. Beckenbach (Princeton, N. J.).

Ellis, H. W. Darboux properties and applications to non-absolutely convergent integrals. Canadian J. Math. 3, 471–485 (1951).

A function $F(x)$ is said to have Darboux properties D, D_k $(0<k\leqq 1)$, D_*, D_d on $[a,b]$ if the following conditions are satisfied for every interval $[l,m]$, $a\leqq l<m\leqq b$, where μ^* denotes outer Lebesgue measure, $[l,m]_y$ the interval $[F(l),F(m)]$, and $F[l,m]$ the F-image of the interval $[l,m]$: (D) $F[l,m]\supset[l,m]_y$;

$$(D_k)\ \nu(l,m)=\mu^*\{F[l,m]\cdot[l,m]_y\}\geqq k|[l,m]_y|;$$

(D_*) $\nu(l,m)>0$ if $F(l)\neq F(m)$; (D_d) $F[l,m]$ is dense in $[l,m]_y$. $F(x)$ has property D' if it has D and if the values y for which $F^{-1}(y)$ is countably infinite and dense (in the order sense) form a null set. $F(x)$ is $[CG]$ (or $[ACG]$) on a set E if $E=\sum_1^\infty E_n$, where each E_n is closed and F is continuous (absolutely continuous) on E_n. Several results concerning Darboux properties are given, for instance: D_k implies D_1. If $F(x)$ is $[CG]$ on $[a,b]$ then properties D', D, D_k, D_d are equivalent and $F+G$ has property D if F has property D and G is continuous. If $F(x)$ is $[ACG]$, has property D and the approximate derivative $ADF\geqq 0$ a.e. on $[a,b]$, then $F(x)$ is non-decreasing. These results are applied to prove that any linear class of functions $F(x)$ which have property D and are $[ACG]$ on $[a,b]$ is a class of generalized indefinite integrals which is compatible with the class of Denjoy integrals and has the properties of uniqueness and order: $ADF\geqq ADG$ a.e. implies $F(b)-F(a)\geqq G(b)-G(a)$.

M. Cotlar (Chicago, Ill.).

Farah, Edison. On an inequality between the L-integrals of functions with complex values. Bol. Soc. Mat. São Paulo 3, no. 1–2 (1948), 31–36 (1951). (Portuguese)

There is given a proof of the inequality

$$\left|\int_E f dx\right|\leqq\int_E |f|dx$$

for Lebesgue integrals over measurable sets E in R^n, and for complex-valued functions $f(x)$. *E. F. Beckenbach.*

Burkill, J. C. On the differentiability of multiple integrals. J. London Math. Soc. 26, 244–249 (1951).

Cette note contient une nouvelle démonstration du théorème suivant, dû à Jessen, Marcinkiewicz et Zygmund [Fund. Math. 25, 217–234 (1935)]: Dans l'espace numérique de dimension k, le système obtenu en associant à chaque point P toutes les cellules de dimension k et d'arêtes parallèles aux axes dérive p.p. toute fonction f telle que $|f|(\log^+|f|)^{k-1}$ soit sommable dans un voisinage de P. La présente démonstration est intéressante car elle s'appuie sur un lemme qui met en évidence la signification de l'hypothèse. Le lemme s'énonce ainsi pour $k=2$: Soient $E_1,\cdots,E_n$ des ensembles disjoints tous contenus dans le carré unité, $p_1,\cdots,p_n$ des poids respectifs associés, σ_α l'ensemble des points des rectangles I tels que $\sum_1^n p_i|E_iI|>\alpha|I|$; on a alors

$$|\sigma_\alpha|<A\sum(p_i\alpha^{-1}\log Ap_i\alpha^{-1})|E_i|+A|\sum E_i|^{\frac{1}{2}}$$

(A désigne des constantes absolues, $|E|$ est la mesure de E).

Il est rappelé sans démonstration que l'hypothèse du théorème est la meilleure possible, comme l'ont démontré les auteurs. *R. de Possel* (Alger).

Zalgaller, V. A. The variation of curves along a fixed direction. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 463–476 (1951). (Russian)

Let $L(C)$ denote the length of a rectifiable curve C in Euclidean n-space, and let $V(e)=V(e, C)$ denote the length (or total variation) of the projection of C on the direction e, where e is any unit vector. Further let $P(C)$ denote the set of points each of which is expressible as an integral from 0 to $L(C)$ of the form $\int f(s)e(s)ds$ where $e(s)$ is a unit vector along the tangent and where $f(s)$ is a scalar measurable function with modulus $\leqq 1$. The author proves (i) that $\int V(e, c)dm(e)=2M\,L(C)$ where $dm(e)$ is the natural measure on the unit sphere and where M is the $(n-1)$-dimensional measure of the solid $(n-1)$-dimensional unit sphere. He shows further (ii) that $P(C)$ is a convex figure with centre at the origin and with $V(e)$ as its function of support (for unit vectors e). The rest of the paper is concerned with additional remarks of a subsidiary nature. The most interesting of these is that in the plane, if two curves C and C^* are such that $V(e, c)\leqq V(e, C^*)$ for every e, then C is limit of a sequence of curves C_n such that $V(e, C_n)\to V(e, C^*)$, whereas in higher space this is no longer the case.

L. C. Young (Madison, Wis.).

Conti, Roberto. Funzioni a variazione limitata in più variabili, nel senso di Fréchet e nel senso di Faedo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 462–467 (1951).

The author proves that the class F of functions, of two or more real variables, of bounded variation (b.v.) in Fréchet's sense is the same as the class F_1 of functions of b.v. in Faedo's sense [S. Faedo, Ann. Scuola Norm. Super. Pisa (2) **6**, 225–246 (1937)]. He notes that the relation $F\subseteq F_1$ is also obtainable from a result given by M. Morse and W. Transue [Canadian J. Math. **1**, 153–165 (1949), Lemma 3.4; these Rev. **10**, 601]. For the definition of F see the review just cited: Faedo's definition is simpler, requiring merely, in the case of a function $f(x, y)$ defined on the rectangle $[a, b; c, d]$, that the sums $\sum_{i=1}^{m}\sum_{j=1}^{n}(-1)^{i+j}f(x_i, y_j)$ should be bounded for all divisions of $[a, b]$ by $\{x_i\}$ with $a=x_0<x_1<\cdots<x_m=b$ and of $[c, d]$ by $\{y_j\}$ with $c=y_0<y_1<\cdots<y_n=d$, where m and n are odd.

H. P. Mulholland (Birmingham).

Mizohata, Sigeru. On Stokes' theorem. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **26**, 175–183 (1951).

Let $F(x, y, z)$, as well as the partial derivatives F_x, F_y, be continuous in an open set containing a surface S with boundary C. According to the theorem of Stokes, the contour integral $\int Fdx$ taken around C can be expressed as a surface integral taken over S in a well-known manner. This paper is a contribution to the extensive literature aiming at establishing the theorem of Stokes under more and more general conditions as regards the surface S. A detailed exposition of the concepts used by the author would be of prohibitive length, but the general idea is to state the conditions upon S in terms of concepts which arose in the modern theory of surface area. *T. Radó.*

Whitney, Hassler. On totally differentiable and smooth functions. Pacific J. Math. **1**, 143–159 (1951).

A number of theorems are proved akin to a result of H. Federer [Trans. Amer. Math. Soc. **55**, 438–456 (1944); these Rev. **6**, 45] which states that, if $f(x)$ is totally differentiable p.p. in a bounded set $P\subset E^n$ ($=n$-dimensional Euclidean space) then there is a function $g(x)$ which is smooth ($=$continuously differentiable) in E^n, and $f(x)=g(x)$ for $x\in Q$ where $|P-Q|<\epsilon$. The author shows that the hypothesis of approximate total differentiability of $f(x)$ is sufficient, and obtains some information concerning the points which may be included in Q. Further theorems establish similar results concerning functions satisfying Lipschitz conditions in P, and concerning functions whose partial derivatives of order m are totally differentiable almost everywhere in an open set P; the latter result generalises a theorem already known [J. Marcinkiewicz, Fund. Math. **27**, 38–69 (1936)] for $n=1$. *U. S. Haslam-Jones.*

* **Whitney, Hassler. r-dimensional integration in n-space.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 245–256. Amer. Math. Soc., Providence, R. I., 1952.

This paper is a concise version of an invited address presented at the International Congress of Mathematicians. The purpose is to propose a general approach to the theory of integration over domains of dimension less than that of the containing space, where the integral is to be thought of as a function of the domain of integration. Historically, the motivation for this line of thought originated in the mechanics of continua, where one operates with concepts like flux, pressure, stress, and the like. In such cases, some sort of "integrand" is a priori given, and by a Riemann-type process an "integral" is then associated with every curve, or every surface, or every solid, depending upon the particular situation. From the analytic side, integration with respect to differential forms yields further motivation for seeking a general theory, and from the formal point of view, the elementary duality relations between homology and cohomology serve as a model, according to patterns now generally used. The importance of this address should be evident to specialists in this difficult and fundamental field.

T. Radó (Columbus, Ohio).

Yamaguti, Masaya. Note on the surface area and the mapping of bounded variation. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **26**, 159–165 (1951).

Let $A: x=x(u, v)$, $y=y(u, v)$ be a continuous mapping from a closed region R in the $w=u+iv$ plane into the $z=x+iy$ plane. One of the purposes of this paper is to compare the concept eBV (essentially of bounded variation) with the concept BVO (essentially of bounded variation in the sense of Okamura) for such mappings. For the concept eBV, the reader may consult the reviewer's book on Length and Area [Amer. Math. Soc. Colloq. Publ., vol. 30, New York, 1948; these Rev. **9**, 505]. The concept BVO is defined, in a plausible manner, in terms of the topological index, but the concept is restricted to special mappings satisfying the following condition: the region R can be subdivided, for every $\epsilon>0$, into a finite number of closed regions $r_1, \cdots, r_j$ of diameter less than ϵ, and having no common interior points, such that the images of the frontiers of these regions have Lebesgue measure zero. The author shows that for such mappings the concepts eBV and BVO are equivalent. Now any definition of bounded variation, for continuous mappings A as above, gives rise to a corresponding definition of surface area, according to a standard pattern (see for example the definition of "lower area" in the book cited above). The author proves the equivalence of the surface areas corresponding to the concepts eBV and BVO. *T. Radó.*

Youngs, J. W. T. The representation problem for Fréchet surfaces. Mem. Amer. Math. Soc., no. 8, i+143 pp. (1951). $1.80.

Let X denote any Peano space, Y a metric space, $\rho(y, y')$ the distance function in Y. Let $f: X \to Y$ denote a single-valued continuous mapping of X into Y, and $f: X \approx Y$ any homeomorphism. Given any two mappings $f_i: X_i \to Y$, $i=1, 2$, into the same space Y, then f_1 and f_2 are called Fréchet equivalent if for every $\epsilon>0$ there is a homeomorphism $h_\epsilon: X_1 \approx X_2$, such that $\rho(f_1x, f_2h_\epsilon x)<\epsilon$ for any $x \in X_1$. This relation, which is metric in character, has all properties of an equivalence and is denoted by $f_1 \sim f_2$. It is possible to divide all mappings f into maximal classes $[f]$ of equivalent mappings. Then each class $[f]$ is called a Fréchet variety V (a Fréchet surface if X is a 2-manifold), and the elements f of the class $[f]$ are all possible representations of the variety V. By representation problem is meant the problem of an intrinsic topological characterization of the representations of a given variety V, that is, to give necessary and sufficient conditions (F-criteria) in order that any two mappings f_1, f_2 are Fréchet equivalent. The importance of the Fréchet equivalence and the difficulty of the representation problem have been recently stressed in numerous papers on surface area and surface integrals. The author solves here the representation problem for all mappings from 2-manifolds. The four basic cases (closed or with boundary, oriented or not oriented 2-manifolds) are considered in a remarkable, unitary way.

Given a mapping $f: X \to Y$ and a subset $A \subset X$, $f|A$ denotes the mapping defined by f on A. $f: X \to Y$ is called monotone if, for any $y \in Y$, the set $f^{-1}y \subset X$ is connected; f is called light if for any $y \in Y$ $f^{-1}y$ is completely disconnected. The following necessary condition (N) in order that $f_1 \sim f_2$, though well known, is by no means obvious: (N) there exists a monotone-light factorization $f_i = lm_i$, $m_i: X_i \to \mathfrak{X}$, $l: \mathfrak{X} \to Y$, $i=1, 2$, where m_i are monotone mappings, l a light mapping and $\mathfrak{X}$ an auxiliary Peano space (middle space). The condition (N) is necessary, but not sufficient for Fréchet equivalence as the author has proved [Ann. of Math. (2) **45**, 753–785 (1944); these Rev. **6**, 278] and a necessary and sufficient condition is that (N) holds and $m_1 \sim m_2$ (reduction theorem) [J. W. T. Youngs, Proc. Nat. Acad. Sci. U. S. A. **32**, 328–330 (1946); these Rev. **8**, 259, 708]. This statement shows that F-criteria can be expressed in terms of monotone mappings.

A set $\mathfrak{X}$ is called a mantoid if $\mathfrak{X}$ is the monotone image of a 2-manifold. A subset U of a mantoid is called a normal region if U is an open connected non-vacuous set whose boundary in $\mathfrak{X}$ has a finite number of components γ and each γ is either a simple point of $\bar{U}$, or a closed Jordan curve and in this case the true cyclic element of $\bar{U}$ containing γ is a 2-cell. The Čech 2-dimensional Abelian group associated with the set X (i.e. to the pair $(X, 0)$, 0 the vacuous set) is denoted by $H^2(X)$. If $f: X \to Y$ is a given mapping, then $f^*: H^2(Y) \to H^2(X)$ is the homomorphism of the Čech theory induced by f. Whenever the homomorphism f^* is a isomorphism from $H^2(Y)$ onto $H^2(X)$ the notation $f^*: H^2(Y) \approx H^2(X)$ is used. With the few definitions above we can state one of the numerous F-criteria proved by the author. (I) Let X_1, X_2 be closed 2-manifolds and $f_1: X_1 \to Y$, $f_2: X_2 \to Y$ any two mappings. Then $f_1 \sim f_2$ if and only if (N) holds and, if U is any normal region of the middle space $\mathfrak{X}$ and $U_i = m_i^{-1}(U)$, $\underset{\sim}{m}_i = m_i | U_i$, $i=1, 2$, then there exists a homeomorphism $h: U_1 \approx U_2$ such that

$$\underset{\sim}{m}_1{}^* = h^* \underset{\sim}{m}_2{}^*, \quad \underset{\sim}{m}_2{}^* = (h^*)^{-1} \underset{\sim}{m}_1{}^*,$$

where $\underset{\sim}{m}_i{}^*: H^2(U) \to H^2(U_i)$, $i=1, 2$, $h^*: H^2(U_2) \approx H^2(U_1)$. Analogous statements hold for mappings from 2-manifolds with boundary, or from oriented and closed 2-manifolds, or from oriented 2-manifolds with boundary. Both analytic and algebraic topology are made use of in the treatment of this difficult matter: in particular, the Roberts and Steenrod characterization of mantoids [Ann. of Math. (2) **39**, 851–862 (1938)], results on the approximation of monotone mappings by means of homeomorphisms [J. W. T. Youngs, Duke Math. J. **15**, 87–94 (1948); these Rev. **9**, 524], the Čech cohomology theory, and the Lefschetz locally compact cohomology theory. *L. Cesari* (Lafayette, Ind.).

Diliberto, S. P., and Straus, E. G. On the approximation of a function of several variables by the sum of functions of fewer variables. Pacific J. Math. **1**, 195–210 (1951).

Il s'agit de l'approximation d'une fonction continue $z(x, y)$ $(0 \leqq x \leqq 1, 0 \leqq y \leqq 1)$ par une somme de la forme $f(x)+g(y)$, où f et g sont continues, et de l'évaluation de la borne inférieure de l'erreur, soit $\mu(z)$, lorsque f et g varient. La recherche de $\mu(z)$ est rattachée à la considération de la fonctionnelle définie à partir des rectangles $(x_i, y_i; i, j=1, 2)$ par $z(x_1, y_1)-z(x_1, y_2)-z(x_2, y_1)+z(x_2, y_2)$ et prolongée dans l'ensemble des polygones fermés à côtés parallèles aux axes. On montre l'existence d'au moins une paire de fonctions f et g, telles que $\sup|z-f-g| = \mu(z)$. On envisage diverses généralisations, soit aux fonctions bornées, soit aux fonctions de plus de deux variables. *J. Favard* (Paris).

Theory of Functions of Complex Variables

***Privalov, I. I. Vvedenie v teoriyu funkcii kompleksnogo peremennogo. [Introduction to the Theory of Functions of a Complex Variable].** 8th ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1948. 451 pp.

This is a carefully written and detailed textbook which covers all the material customary in an introductory course and goes as far as the strong form of Cauchy's theorem, Picard's theorem via Bloch's theorem, conformal mapping on the boundary, and classical results on univalent functions. *R. P. Boas, Jr.* (Evanston, Ill.).

Kuramochi, Zenjiro. On sufficient conditions for a function to be holomorphic in a domain. Osaka Math. J. **3**, 21–47 (1951).

The author considers generalizations of Menchoff's theorems [see, for instance, D. Menchoff, Les conditions de monogénéité, Actualités Sci. Ind., no. 329, Hermann, Paris, 1936], which give various sufficient conditions for a continuous function $f(z)$ of a complex variable z, defined in a region R, to be holomorphic there. Many of these conditions include the assumption that $f(z)$ shall be univalent in R. In the present paper similar conditions, but without the assumption of univalence, are studied. *W. Seidel.*

Popov, I. V. On a question of Prof. V. N. Deputatov. Uspehi Matem. Nauk (N.S.) **6**, no. 4(44), 170–171 (1951). (Russian)

The question is whether all elementary functions are regular in all their branches outside some countable set. The author shows that this is not the case by proving that $[\ln(z-a)+\alpha \ln(z-b)+i \ln(z-c)+i\beta \ln(z-d)]^{-1}$, α, β irrational, has singularities which are everywhere dense. *P. Davis* (Cambridge, Mass.).

Evgrafov, M. A. Power series with integral coefficients. I. Mat. Sbornik N.S. **28(70)**, 715–722 (1951). (Russian)

The author generalizes a theorem of Szegö on power series with integer coefficients. It is shown that if $f(z)=\sum_{n=0}^{\infty}a_nz^n$, a_n integral, is single valued in $|z|<1$ and with the exception of a finite number of points is regular there, and if, in addition, $f(z)$ can be continued analytically across some arc of the unit circle, then it is a rational function. *P. Davis.*

Evgrafov, M. A. Power series with integer coefficients. II. Mat. Sbornik N.S. **29(71)**, 121–132 (1951). (Russian)

The following result is established, again generalizing a theorem of Szegö on power series with integer coefficients. Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$, a_n integral. Let $P(z)$ be a polynomial and $f(z)P(z)=\sum_{n=0}^{\infty}a_n'z^n$, $F_1(z)=\sum_{n=0}^{\infty}a_n'z^n/n!$. The following conditions now suffice to insure the rationality of $f(z)$:

1) $|F_1(re^{i\theta})|<Me^{r(1+\delta(r))}$, 2) $|F_1(re^{i\theta})|<Me^{r(1-\epsilon(r))}$

for $|\theta-\theta_0|\leqq\eta$, 3) $\epsilon(r)\to 0$ for $r\to\infty$, 4) $\delta(r)/\epsilon(r)\to 0$ for $r\to\infty$, 5) $r\epsilon(r)/\ln r\to\infty$ for $r\to\infty$. By way of showing the exactness of his conditions, the author has constructed a counterexample $f(z)=\sum_{n=0}^{\infty}a_nz^n$, a_n integral, $f(z)$ not rational, such that if $F(z)=\sum_{n=0}^{\infty}a_nz^n/n!$, then $|F(re^{i\theta})|<Me^{r(1+\delta(r))}$, and for arbitrary $\alpha>0$, an $r_0=r_0(\alpha)$ and an $\eta=\eta(\alpha)$ may be found such that $|F(re^{i\theta})|<Me^{r(1-e^{-\alpha}\delta(r))}$ for $|\theta|<\eta$ and $r>r_0$. Moreover, such a counterexample may be found for all $\delta(r)$ satisfying the above conditions. *P. Davis.*

Havinson, S. Ya. An estimate of Taylor sums of bounded analytic functions in a circle. Doklady Akad. Nauk SSSR (N.S.) **80**, 333–336 (1951). (Russian)

Let $f(z)$ be regular in the unit circle, $|f(z)|\leqq 1$. The author gives the precise asymptotic estimates for the least upper bound, for all such $f(z)$, of the absolute values of the partial sums $S^{n,a}(f)$ of the Taylor series of $f(z)$ about the (complex) point a, $0<|a|<1$, evaluated at the (complex) point x, $|x|\leqq 1$. For $a=0$ the asymptotic value $\pi^{-1}\log n$ was given by Landau [cf. Macintyre and Rogosinski, Acta Math. **82**, 275–325 (1950); these Rev. **12**, 89]. For $x\neq 1$, the asymptotic value is

$$(2\pi an)^{-1}\frac{1-a}{|1-x|}\left\{\frac{|x-a|}{1-a}\right\}^n,$$

a (previously unpublished) result of A. Markuševič. For $x=1$ it is $(2\pi)^{-1}\log n$, i.e. half the value for $a=0$.

R. P. Boas, Jr. (Evanston, Ill.).

Erdös, P., Herzog, F., and Piranian, G. Schlicht Taylor series whose convergence on the unit circle is uniform but not absolute. Pacific J. Math. **1**, 75–82 (1951).

Two examples of Taylor series $\sum a_kz^{m_k}$ that are schlicht in $|z|\leqq 1$, and that converge uniformly but not absolutely on $|z|=1$. In the first example there is exactly one singular point on $|z|=1$, while in the second $m_{k+1}-m_k\to\infty$. The point, in the second example, is that "schlichtness" implies the divergence of $\sum 1/m_k$, so that the m_k cannot grow too rapidly. *W. W. Rogosinski* (Newcastle upon Tyne).

Yu, Chia-Yung. Remarques sur une inégalité fondamentale de M. Mandelbrojt. Bull. Sci. Math. (2) **75**, 107–113 (1951).

The author generalized Mandelbrojt's "fundamental inequality" [Ann. Sci. École Norm. Sup. (3) **63**, 351–378 (1946); these Rev. **9**, 229, 735] by introducing a different auxiliary function. The essential point is to replace $\prod_{m\neq n}\lambda_m^2/|\lambda_n^2-\lambda_m^2|$ in the conclusion by

$$\prod_{|\lambda_m-\lambda_n|\geq 1}\frac{\lambda_m^2}{|\lambda_m^2-\lambda_n^2|}\prod_{|\lambda_m-\lambda_n|<1}\frac{\lambda_m}{|\lambda_m-\lambda_n|},\qquad m\neq n.$$

This allows a weaker separation condition on $\{\lambda_n\}$ to be used in such applications to Dirichlet series as are given by Mandelbrojt [Rice Inst. Pamphlet **31**, 159–272 (1944); these Rev. **6**, 267] and Brunk [Bull. Amer. Math. Soc. **55**, 204–212 (1949); these Rev. **10**, 436]. Detailed statements are given of several theorems on the linear order of a function defined by a Dirichlet series. Further generalizations are indicated. *R. P. Boas, Jr.* (Evanston, Ill.).

Mergelyan, S. N. On an integral connected with analytic functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 395–400 (1951). (Russian)

The author constructs a function which is bounded and analytic in $|z|<1$ and for which $\iint|f'(z)|dS$ extended over $|z|<1$ is infinite, and discusses some consequences. In particular, the maximum of the corresponding integral for the class of polynomials of degree at most n which are bounded by 1 must become infinite as $n\to\infty$; there exists a three-dimensional set which is the Cartesian product of a linear and a plane set, over which the principal-value integral

$$\iiint\frac{x(y-z)}{[x^2+(y-z)^2]^2}dxdydz$$

diverges. *R. P. Boas, Jr.* (Evanston, Ill.).

Walsh, J. L. On Rouché's theorem and the integral-square measure of approximation. Proc. Amer. Math. Soc. **2**, 671–681 (1951).

Comme conséquence du théorème de Rouché et de l'intégrale de Cauchy, l'auteur remarque le résultat suivant: Soit R une région de frontière rectifiable B, à l'intérieur de laquelle $f(z)$ est holomorphe, R_1 une région intérieure à R ainsi que sa frontière B_1; on suppose que $f(z)\neq 0$ sur B_1; alors il existe un nombre positif $\delta(R,R_1,f)$ tel que toute fonction $F(z)$, holomorphe dans R et telle que

$$(f,F)=\frac{1}{2\pi}\int_B|f-F|^2|dz|<\delta,$$

a dans R_1 le même nombre de zeros que f.

L'auteur détermine δ de façon précise dans des cas simples; par example, dans le cas où B est le cercle de rayon 1, si $f=\sum_0^\infty a_nz^n$, avec $\sum_0^\infty|a_n|^2$ convergente, et si pour n donné, entier >0,

$$[f(z),z^n]<\epsilon<n^n/(n+1)^{n+1}$$

alors $f(z)$ a n zéros dans $|z|<r_1$ et n'a pas de zéro dans $r_1\leqq|z|\leqq r_2$, r_1 et r_2 étant les racines positives de $r^{2n}(1-r^2)=\epsilon$. (Le cas $n=0$ est traité à part.) L'auteur retrouve ainsi des résultats de Petrovitch [Bull. Soc. Math. France **29**, 303–312 (1901)] complétés par Landau [Tôhoku Math. J. **5**, 97–116 (1914)]; il obtient de nouveaux résultats sur les zéros des polynomes et pose des problèmes en liaison avec la question traitée. *J. Favard* (Paris).

Bose, S. K. On the maximum modulus of an integral function. Bull. Calcutta Math. Soc. **43**, 25–26 (1951).

The inequality $\limsup\{\log\mu(r)\}/\{\nu(r)\log r\}\leqq 1$, where $\mu(r)$ is the maximum term, and $\nu(r)$ is the rank of this term, in the power series of an entire function of finite order, is an immediate consequence of an inequality of Valiron [Lec-

tures on the general theory of integral functions, Toulouse, 1923, p. 31]. The author observes this fact and notes some corollaries. *R. P. Boas, Jr.* (Evanston, Ill.).

Noble, M. E. Non-measurable interpolation sets. I. Integral functions. Proc. Cambridge Philos. Soc. **47**, 713–732 (1951).

Noble, M. E. Non-measurable interpolation sets. II. Functions regular in an angle. Proc. Cambridge Philos. Soc. **47**, 733–740 (1951).

The author's problem in the first paper is to give conditions on a sequence $\{z_n\}$ so that an entire function $f(z)$ of order ρ will have its rate of growth over the plane determined by its growth over $\{z_n\}$. The precise relationship of the author's results to earlier work of others is discussed in the papers. The chief improvement in the author's results over those of Pfluger [Comment. Math. Helv. **14**, 314–349 (1942); **18**, 177–203 (1946); these Rev. **5**, 258; **7**, 426] is that the author relaxes the condition of "measurability" of $\{z_n\}$ (essentially that the points have a density in every sector), replacing it by hypotheses on the Pólya maximum density and on a kind of minimum density less restrictive than Pólya's; but he imposes the condition $n(r, S_1)/n(r, S_2) \to \alpha_1/\alpha_2$, where $n(r, S)$ is the number of z_n of absolute value not exceeding r in the sector S of opening α, a condition which makes the distribution of the z_n substantially the same in all directions. The main part of the paper is devoted to appropriate estimates for a canonical product over the z_n, and from these the author derives theorems estimating $\limsup r^{-\rho} \log M(r)$ in terms of $\limsup |z_n|^{-\rho} \log |f(z_n)|$ and also the theorem that $f(z)$ is constant if $|f(z_n)|$ is bounded and the type of $f(z)$ is small enough. The detailed statements of the theorems are too long to reproduce here.

In the second paper the main theorem of the first paper is used to prove growth theorems and uniqueness theorems for functions which are of finite order in an angle. There is a simplification resulting from the fact that integral orders no longer need special treatment. The results are significant generalizations of previous work, to which full references are given. *R. P. Boas, Jr.* (Evanston, Ill.).

Naftalevič, A. G. Some questions of the interpolation of meromorphic functions. Doklady Akad. Nauk SSSR (N.S.) **80**, 329–332 (1951). (Russian)

The author announces generalizations of results of J. M. Whittaker [Proc. London Math. Soc. (2) **40**, 255–272 (1935)] and Z. Mursi [Bull. Sci. Math. (2) **73**, 96–112 (1949); these Rev. **11**, 340] on constructing meromorphic or entire functions having a group of terms of their Laurent expansions prescribed at each point of an infinite sequence. The principal novelty is that the author estimates the type as well as the order of the functions. *R. P. Boas, Jr.*

Shah, S. M. Some theorems on meromorphic functions. Proc. Amer. Math. Soc. **2**, 694–698 (1951).

L'auteur étend des théorèmes sur les fonctions $F(z)$ méromorphes et d'ordre fini ρ, donnés par Okada [mêmes Proc. **1**, 246–249 (1950); ces Rev. **11**, 719]: Okada se bornait au cas où $F(z)$ est le quotient de deux produits canoniques, Shah traite le cas général. Il utilise ses travaux précédents [J. London Math. Soc. **15**, 23–31 (1940); J. Indian Math. Soc. (N.S.) **5**, 179–188 (1941); ces Rev. **1**, 307, 400; **4**, 6] et les résultats du livre de Nevanlinna [. . . Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Gauthier-Villars, Paris, 1929]. Il désigne par $I(r)$ la valeur moyenne du module maximum $r^{-1}\int_0^r \log^+ M(t, F)dt$ et par $n(r, \infty)$ et $n(r, 0)$ le nombre des pôles et des zéros dans le cercle de rayon r. Il montre que si ρ est positif et non entier,

$$\liminf_{r\to\infty} I(r)/[n(r, \infty)+n(r, 0)] < \infty. \tag{1}$$

Dans tous les cas où $\rho > 0$, il donne également une inégalité valable lorsque le degré du facteur exponentiel figurant dans $F(z)$ est inférieur ou égal aux genres des produits canoniques formés avec les pôles et les zéros. Lorsque $\rho = 0$, il montre que

$$\liminf_{r\to\infty} I(r)/[N(r, \infty)+N(r, 0)] < \infty, \tag{2}$$

N étant la fonction de Jensen. Des exemples relatifs aux fonctions entières montrent que les résultats ne peuvent guère être améliorés. [Remarque du référent. Les propriétés telles que (1) doivent être rapprochées des théorèmes des pages 25 et 51 du livre de Nevanlinna; il semble d'après mes résultats [Proc. Internat. Cong. Math., Cambridge, Mass., 1950, vol. I, pp. 404–405, Amer. Math. Soc., Providence, R. I., 1952] que dans (2), la somme $N(r, \infty)+N(r, 0)$ puisse être remplacée par le plus grand des nombres $N(r, \infty)$ et $N(r, 0)$.] *G. Valiron* (Paris).

Ilieff, Ljubomir. Über die Abschnitte der 3-symmetrischen schlichten Funktionen. C. R. Acad. Bulgare Sci. **3**, no. 1 (1950), 9–12 (1951). (German. Russian summary)

The author proves a slightly weakened version of one of his earlier results [Doklady Akad. Nauk SSSR (N.S.) **79**, 9–11 (1951); these Rev. **13**, 123]. *A. W. Goodman.*

Tims, S. R. A theorem on functions schlicht in convex domains. Proc. London Math. Soc. (3) **1**, 200–205 (1951).

Noshiro [J. Fac. Sci. Hokkaido Univ. Ser. I. **2**, 129–155 (1934)] a démontré que, si $\Phi(z)$ est régulière dans le domaine simplement connexe et convexe D et que la partie réele de $\Phi'(z)$ dérivée de $\Phi(z)$, ne s'annule pas dans D, la fonction $\Phi(z)$ est univalente dans D. L'auteur montre que pour tout domaine D non-convexe, simplement connexe, il existe une fonction $\Phi(z)$ régulière dans D, dont la dérivée a sa partie réelle différente de zéro dans D, et qui n'est pas univalente dans D. *S. Stoïlow* (Bucarest).

Ohtsuka, Makoto. On the cluster sets of analytic functions in a Jordan domain. J. Math. Soc. Japan **2**, 1–15 (1950).

Let D be a region in the z-plane bounded by a Jordan curve C, z_0 a point of C, and $f(z)$ a meromorphic function in D. The intersections with the circle $|z-z_0| < r$ of D and C are denoted by D_r and C_r, respectively. Let V_r denote the set of values assumed by $f(z)$ in D_r. The intersection $\bigcap_{r>0} \bar{V}_r = S_{z_0}^{(D)}$ is called the cluster set of $f(z)$ in D at z_0, and $\bigcap_{r>0} V_r = R_{z_0}^{(D)}$ the range of values of $f(z)$ in D at z_0. Similarly, if $M_r^{(C)}$ denotes the union $\bigcup S_{z'}^{(D)}$ taken over all z' in C_r, different from z_0, then the intersection $\bigcap_{r>0} \bar{M}_r^{(C)} = S_{z_0}^{(C)}$ is called the cluster set of $f(z)$ on C at z_0. If for a Jordan arc L in D terminating in z_0, $\lim f(z) = \alpha$ exists as z approaches z_0 on L, α is called an asymptotic value of $f(z)$ at z_0. The set of all asymptotic values of $f(z)$ at z_0 is denoted by $\Gamma_{z_0}^{(D)}$ and is called the convergence set of $f(z)$ at z_0. The intersection $\bigcap_{r>0} \bar{Y}_r^{(C)}$, where $Y_r^{(C)} = \bigcup \Gamma_{z'}^{(D)}$ taken over all z' in C_r, different from z_0, is denoted by $\Gamma_{z_0}^{(C)}$. Numerous relations are found for the sets $S_{z_0}^{(D)}$, $S_{z_0}^{(C)}$, $\Gamma_{z_0}^{(C)}$, $R_{z_0}^{(D)}$, as well as for some other analogous sets. It is shown, for instance, that all values in $S_{z_0}^{(D)} - \Gamma_{z_0}^{(C)}$, except perhaps two, are contained in $R_{z_0}^{(D)}$. Certain older results of Gross, Iversen, Cartwright,

Seidel, and Hössjer are either proved anew or are extended. There is a detailed discussion of the special case when $f(z)$ is regular and bounded in D. *W. Seidel.*

Kaplan, Wilfred. On Gross's star theorem, schlicht functions, logarithmic potentials and Fourier series. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 86, 23 pp. (1951).

Ce mémoire montre l'importance en représentation conforme des ensembles de capacité nulle de la frontière. Dans ce but, il étudie d'abord le rôle joué par ces ensembles vis-à-vis du potentiel logarithmique d'une masse répartie sur un cercle, et de la convergence de la série trigonométrique conjuguée d'une fonction à variation bornée. Enfin, un paragraphe assez indépendant est consacré à des extensions du théorème de l'étoile de Gross, relatif à la surface de Riemann de l'inverse d'une fonction méromorphe, ainsi qu'à des applications de ces extensions.

Résumons les principaux résultats. I. Soit $\varphi(z)$ une fonction univalente dans le cercle C: $|z|=1$. L'ensemble E des points z_0 de C tels que $\lim_{z\to z_0}\varphi(z)=\infty$ est de capacité nulle. Inversement, tout ensemble fermé de capacité nulle peut être obtenu de cette manière. II. La série trigonométrique conjuguée d'une fonction $f(\theta)$ à variation bornée, périodique, diverge en tout point d'un ensemble E de capacité nulle et converge partout ailleurs. Inversement, tout ensemble fermé de capacité nulle peut être obtenu de cette manière.

Les démonstrations s'appuient essentiellement: 1) sur des résultats de Salem et Zygmund [Trans. Amer. Math. Soc. 59, 23–41 (1946); ces Rev. 7, 434]; 2) sur l'identité, à une constante multiplicative près, entre les fonctions qui représentent conformément $|z|<1$ sur un domaine étoilé par rapport à l'∞ dans une direction donnée, et les fonctions dont la partie imaginaire est le potentiel logarithmique d'une distribution de masses positives sur le cercle; 3) sur le théorème suivant (th. 1 du mémoire) que démontre l'auteur en utilisant l'inégalité de Schwarz: Si $\varphi(z)$ est univalente dans le domaine D: $0<x<1$, $0<y<h(x)$, où $h(x)$ est semi-continue inférieurement et vérifie $1\leqq h(x)\leqq+\infty$, alors, l'ensemble des x pour lesquels on a $h(x)<\infty$ et $\lim_{z\to h(x)}\varphi(x+iy)=0$ est de mesure nulle.

C'ést à partir de ce théorème qu'est obtenue l'extension du théorème de Gross. Celui-ci affirme qu'un élément de l'inverse d'une fonction méromorphe peut être prolongé à partir de son centre le long de tout rayon jusqu'à l'∞, à l'exception d'un ensemble de rayons de mesure nulle. L'auteur démontre que les rayons peuvent être remplacés par les images d'une famille de droites parallèles à l'axe imaginaire, données par une fonction méromorphe dans un domaine tel que D (nous désignerons un tel prolongement par P). Il déduit de là plusieurs conséquences relatives à la structure de la surface de Riemann de l'inverse d'une fonction méromorphe.

Enfin, il considère des fonctions multiformes quelconques, et, supposant le prolongement P possible pour deux de ces fonctions, il démontre qu'il l'est encore pour leur somme, leur produit, leur superposition. En particulier, le prolongement est possible pour les fonctions d'automorphisme d'une fonction méromorphe. *R. de Possel* (Alger).

Davis, Philip, and Pollak, Henry. A theorem for kernel functions. Proc. Amer. Math. Soc. 2, 686–690 (1951).

Let D be a schlicht domain and let $K(z,\bar{t})$ be its Bergman kernel function. If $r(z)$ is the shortest distance of a point z ($z\in D$) from the boundary of D, the authors show that $1/r(z)=\limsup_{n\to\infty}(e/n)[K^{(n,n)}(z,\bar{z})]^{1/2n}$, where

$$K^{(m,n)}(z,\bar{t})=\frac{\partial^{m+n}}{\partial z^m\partial\bar{t}^n}K(z,\bar{t}).$$

In the case of a simply-connected domain, $K(z,\bar{t})$ is closely related to the function $f(z)$ mapping D onto the unit circle and the authors use this connection in order to establish a formula for $r(z)$ in terms of the Taylor coefficients of $f(z)$. *Z. Nehari* (St. Louis. Mo.).

Specht, E. J. Estimates on the mapping function and its derivatives in conformal mapping of nearly circular regions. Trans. Amer. Math. Soc. 71, 183–196 (1951).

Soit C une courbe de Jordan d'équation $\rho=\rho(\varphi)$ $(0\leqq\varphi\leqq2\pi)$, satisfaisant à $1\leqq\rho(\varphi)\leqq1+\epsilon$. Soit $f(z)$ la fonction qui représente conformément le cercle $|z|<1$ sur l'intérieur de C et satisfait à $f(0)=0$, $f'(0)>0$. On sait que l'on a: $|f(z)-z|<K\epsilon$ ($K=$cte.) si $\rho(\varphi)$ satisfait à une condition de Lipschitz d'ordre 1 [A. R. Marčenko, C. R. (Doklady) Acad. Sci. URSS (N.S.) 6 (1935 I), 287–290]. L'auteur expose une démonstration de cette propriété et donne une borne pour K; puis il établit des bornes pour les quantités $|f'(z)-1|$, $|f''(z)|$, $|f^{(p)}(z)|$, $p>1$, moyennant des hypothèses de régularité convenables sur $\rho(\varphi)$. La borne trouvée pour $|f^{(n)}(z)|$ dépend des bornes de $|\omega^k(\varphi)|$ pour $k=0,1,2,\cdots,n-1$ et de celle de

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi}\left|\frac{\omega^{(n-1)}(\varphi)-\omega^{(n-1)}(\varphi_0)}{\sin\frac{1}{2}(\varphi-\varphi_0)}\right|d\varphi\quad(0\leqq\varphi_0\leqq2\pi)$$

où l'on a posé $\omega(\varphi)=-\operatorname{arc\,tg}\rho'/\rho$. Les démonstrations utilisent principalement l'intégrale de Poisson, et s'appuient sur certains résultats de S. E. Warschawski établissant la continuité de $f^{(n)}(z)$ pour $|z|=1$ [Math. Z. 35, 321–456 (1932); Trans. Amer. Math. Soc. 38, 310–340 (1935)]. *J. Lelong* (Lille).

Huber, Heinz. Über analytische Abbildungen von Ringgebieten in Ringgebiete. Compositio Math. 9, 161–168 (1951).

Let R_1 and R_2 be two finite doubly-connected schlicht domains whose Riemann moduli are M_1 and M_2, respectively, and let F denote the class of all analytic functions $f(z)$ which are regular and single-valued in R_1 and satisfy $f(z)\in R_2$ if $z\in R_1$. If C is a Jordan curve in R_1 which surrounds the inner boundary of R_1 and w_0 denotes a point on the inner boundary of R_2, then the integer N defined by $2\pi N=\Delta\arg\{f(z)-w_0\}$, where $\Delta\arg\{f(z)-w_0\}$ is the increment of the argument in question if z describes C in the positive sense, is independent of the particular curve C chosen. The author proves that, for the functions $f(z)\in F$, N is contained between $-M_2/M_1$ and M_2/M_1. *Z. Nehari* (St. Louis, Mo.).

Mori, Akira. On conformal representation of multiply connected polygonal domain. J. Math. Soc. Japan 2, 187–197 (1951).

Let D be a schlicht domain bounded by n closed analytic curves and let C be the class of analytic functions $f(z)$ with the following properties: (a) $f(z)$ is of the form $f(z)=(z-z_0)^\alpha g(z)$ [α real, $z_0\in D$, $g(z)\neq0$ in D] and $g(z)$ is regular in D; (b) $|f(z)|$ is single-valued in D; (c) any branch of $\arg f(z)$ is bounded near the boundary Γ of D and $\lim_{z\to\zeta}\arg f(z)=2\pi\sigma(\zeta)$ [$\zeta\in\Gamma$] exists everywhere on Γ except possibly on a countable set and is of bounded variation. The author proves that a necessary and sufficient con-

dition for a function $f(z)$ to belong to C is the existence of the representation $f(z)=\exp \int_\Gamma \log \Omega(z,\zeta)d\sigma(\zeta)$, where the analytic function $\Omega(z,\zeta)$ yields a certain radial slit mapping of D. As an application, the author derives a formula for the function mapping a ring with circular slits onto a domain with polygonal boundaries. *Z. Nehari* (St. Louis, Mo.).

Mori, Akira. Conformal representation of multiply connected domain on many-sheeted disc. J. Math. Soc. Japan **2**, 198–209 (1951).

It is well known that there exist $(1, m)$ conformal mappings of a schlicht domain of connectivity n $(n \leqq m)$ onto the unit circle and that these mappings can be made unique by certain additional conditions. The author gives a new proof of this result along lines which differ somewhat from the usual procedures employed for this purpose.

Z. Nehari (St. Louis, Mo.).

Schiffer, M., and Spencer, D. C. On the conformal mapping of one Riemann surface into another. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **94**, 10 pp. (1951).

Two Riemann surfaces $\mathfrak{M}$ and $\mathfrak{N}$ are given, along with an analytic mapping $\pi=\phi(p)$ of a neighborhood of $p_0 \varepsilon \mathfrak{M}$ into a neighborhood of $\pi_0 \varepsilon \mathfrak{N}$. Necessary and sufficient conditions are obtained for the continuing of $\phi(p)$ over $\mathfrak{M}$ so that $\phi(p)$ gives a one-one mapping between the two entire surfaces. Green's function $G(p, q)$ is defined by means of the closed double $\mathfrak{F}$ of $\mathfrak{M}$ when the latter is open, and through $G(p, q)$ a bilinear differential $L(p, q)=-(2/\pi)\partial^2 G(p, q)/\partial p \partial q$ is defined. It is shown that $-L(p, \tilde{q})$, $\tilde{q} \varepsilon \tilde{\mathfrak{M}}$, the replica of $\mathfrak{M}$ in $\mathfrak{F}$, is the kernel of every complete orthonormal set of differentials on $\mathfrak{M}$, a generalization of the kernel function of S. Bergman for the complex plane [see Bergman and Schiffer, Compositio Math. **8**, 205–249 (1951); these Rev. **12**, 602]. If $\mathfrak{M}$ is closed, $L(p, q)$ is defined as a limiting case of the $L(p, q)$ for open surfaces.

The necessary and sufficient conditions for the mapping through $\pi=\phi(p)$ are

$$\left|\sum_{\mu,\nu=0}^{N} v_{\mu\nu}x_\mu x_\nu\right|^2 \leqq \sum_{\mu,\nu=0}^{N} b_{\mu\nu}x_\mu \bar{x}_\nu \cdot \sum_{\mu,\nu=0}^{N} w_{\mu\nu}x_\mu \bar{x}_\nu,$$

where the coefficients $b_{\mu\nu}$, $w_{\mu\nu}$, $v_{\mu\nu}$ may be computed from the kernels of $\mathfrak{M}$ and $\mathfrak{N}$ if the map $\phi(p)$ is known locally. Here the x_μ, x_ν are N arbitrary complex numbers. Specialization of the domains $\mathfrak{M}$ and $\mathfrak{N}$ leads to classical problems.

M. S. Robertson (New Brunswick, N. J.).

Schiffer, M., and Spencer, D. C. A variational calculus for Riemann surfaces. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **93**, 9 pp. (1951).

A study is made concerning the way in which the Green's function of a finite Riemann surface changes when the surface is altered. A systematic variational calculus is developed, with formulae for the change in the Green's function corresponding to the four basic operations of obtaining a finite Riemann surface from a sphere: (I) cutting out holes; (II) attaching handles; (III) attaching cross-caps; and (IV) attaching cells. *M. S. Robertson.*

Jenkins, James A. On a theorem of Spencer. J. London Math. Soc. **26**, 313–316 (1951).

Simple proof of a theorem of D. C. Spencer [Proc. Nat. Acad. Sci. U. S. A. **26**, 616–621 (1940); these Rev. **2**, 79]. Use is made of a logarithmic metric. Analogous results are indicated. *M. Heins* (Providence, R. I.).

Strebel, Kurt. Eine Ungleichung für extremale Längen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **90**, 8 pp. (1951).

The following inequality for extremal length is established. Let λ_1 and λ_2 denote the respective extremal lengths of the families of curves $\{\gamma_1\}$ and $\{\gamma_2\}$, and let λ denote the extremal length of $\{\gamma_1\} \cup \{\gamma_2\}$. Then $\lambda^{-1} \leqq \lambda_1^{-1}+\lambda_2^{-1}$. This inequality is applied to an extremal problem of Grötzsch [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **84**, 3–14 (1932)] concerning the conformal mapping of certain slit annular regions.

M. Heins (Providence, R. I.).

Tsuji, Masatsugu. Some theorems on open Riemann surfaces. Nagoya Math. J. **3**, 141–145 (1951).

The author employs a method due to T. Kuroda to prove the following theorem: Let F be a Riemann surface with null boundary and let Δ be a region of F which is not relatively compact and has a relative boundary consisting of analytic curves. If u is harmonic in Δ, vanishes continuously on the relative boundary of Δ and has finite Dirichlet integral, then $u \equiv 0$. This result is applied to obtain a maximum principle for functions harmonic in Δ with finite Dirichlet integral. The method of proof of the first of these results is closely related to the method employed by R. Nevanlinna in the study of related questions [Ann. Acad. Sci. Fennicae. Ser. A. no. **45** (1948); these Rev. **10**, 28]. In the second part of the paper, the author studies the Green's functions of relatively compact subregions of a given open Riemann surface with the aid of a method due to the reviewer [Ann. of Math. (2) **50**, 686–690 (1949); these Rev. **11**, 93].

M. Heins (Providence, R. I.).

Ahlfors, Lars V. Remarks on the classification of open Riemann surfaces. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **87**, 8 pp. (1951).

The paper brings two counter-examples. The first shows that the existence of non-trivial bounded analytic functions on a Riemann surface depends on the inner structure of the surface in addition to the nature of the ideal boundary. The second example is an error; the reasoning breaks down because the function $U_1(z)$ on page 5 may become singular at the end points of the slits and therefore is not subharmonic as stated. The question as to the existence of a Riemann surface with a Green's function but no bounded harmonic functions remains open. *L. V. Ahlfors.*

Lokki, Olli. Beiträge zur Theorie der analytischen und harmonischen Funktionen mit endlichem Dirichletintegral. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **92**, 11 pp. (1951).

This paper is concerned with analytic and harmonic functions with finite Dirichlet integral. Part of the results are to be found in a paper of Ahlfors and Beurling [Acta Math. **83**, 101–129 (1950); these Rev. **12**, 171]. Other results are the following: (a) There exist bounded univalent analytic functions on a domain B_D (i.e. one which admits non-constant analytic functions with bounded Dirichlet integral) whose boundary has finite positive linear measure. (b) If a closed set lies on a line and has positive logarithmic capacity but zero span, then every function harmonic in the complementary region with finite Dirichlet integral takes on the same value at points of the region which are symmetric with respect to the line.

M. Heins (Providence, R. I.).

Royden, H. L. Some remarks on open Riemann surfaces. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 85, 8 pp. (1951).

Let O_F be the class of Riemann surfaces R not admitting functions belonging to a given class F. Denote by H and A the classes of harmonic or analytic single-valued non-constant functions. Use B and D as abbreviations of "bounded" or "with a finite Dirichlet integral". The author establishes the relations: (1) $O_{HB} \subseteq O_{HD}$, (2) $O_{HD} = O_{HBD}$, (3) $O_{AB} \subset O_{AD}$. He further proves: (4) The classes O_{HB}, O_{HD} are determined by the ideal boundary. (5) R non-$\in O_{HB}$ if and only if R has two disjoint regions of hyperbolic character. In the proofs, essential use is made of Stokes' theorem and exterior differential calculus.

It is a sign of the present intense investigation of Riemann surfaces that several of the above results have been independently found by other authors. (1) was proved simultaneously by Virtanen [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 75 (1950); these Rev. 12, 403]. (3) was established by Ahlfors and Beurling [Acta Math. 83, 101–129 (1950); these Rev. 12, 171] for planar surfaces; simultaneously with the author's generalization of this result, an independent proof, using Newtonian capacity, was given by Parreau [C. R. Acad. Sci. Paris 231, 751–753 (1950); these Rev. 12, 402]; later (3) was reproved by Lokki [see the preceding review]. (4) was established by the reviewer [C. R. Acad. Sci. Paris 230, 42–44 (1950); these Rev. 11, 342]. For (5), an independent proof was given by Bader and Parreau [C. R. Acad. Sci. Paris 232, 138–139 (1951); these Rev. 12, 603].

For a unification of the ever-growing diversity of notations, it seems to be proper to keep the notation N_F for function-theoretic null sets of Ahlfors and Beurling [loc. cit.] and use the symbol O_F for surface classes with $F=0$ [Ahlfors, see the second preceding review].

L. Sario (Stanford University, Calif.).

Behnke, H. und Stein, K. Die Singularitäten der analytischen Funktionen mehrerer Veränderlichen. Nieuw Arch. Wiskunde (2) 23, 227–242 (1951).

Kiyosi Oka [Tôhoku Math. J. 49, 15–52 (1942); these Rev. 7, 290], using the results of his earlier papers and results of others in the field, gave a complete solution of an important problem in the theory of analytic functions of two complex variables; he proved that every finite schlicht pseudo-convex region in the space of two complex variables is a domain of regularity. In the present paper the authors discuss the historical and mathematical relationship of this result to a number of other results and concepts in the theory of two complex variables, particularly the continuity theorem, Cousin's first problem, and the concepts of regularity envelopes, regular convexity and various types of pseudo-convexity. They bring together these results and concepts in one place and give a proof of Oka's theorem. They conclude the article with an account of several problems which can be solved by use of the theorem.

W. T. Martin (Princeton, N. J.).

* **Mitchell, Josephine. An example of a complete orthonormal system and the kernel function in the geometry of matrices.** Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 155–163. University of Toronto Press, Toronto, 1951. $6.00.

Let $Z=(z_{jk})$ be a non-symmetric m by n matrix, and $\bar{Z}'=(\bar{z}_{kj})$. In the space of mn complex variables, the author studies complete orthonormal systems for the domain $B: I-Z\bar{Z}'>0$ in the sense of the inner product $J(f,g)=\int_B f\bar{g}\hat{Z}$, where $\hat{Z}$ is the Euclidean volume element. By means of a process due to Hua, $J(f,g)$ may be reduced to an mth order iterated integral over $2n$-dimensional manifolds. For 2 by 2 matrices, a complete orthonormal system is derived explicitly, and it is thereby shown that $(1/12\pi^4)[\det(I-Z\bar{Z}')]^{-4}$ may be identified with the Bergman kernel function $K(z,\bar{z})$ for B. This result enables a relation to be established between Bergman's invariant metric for B and a theorem of Hua on the invariance of the quadratic form $\sigma((I-Z\bar{Z}')^{-1}dZ(I-\bar{Z}'Z)^{-1}d\bar{Z}')$.

P. Davis.

Lelong-Ferrand, Jacqueline. Sur une classe de représentations canoniques. C. R. Acad. Sci. Paris 233, 544–546 (1951).

The author continues her work with preholomorphic functions [Bull. Sci. Math. (2) 68, 152–180 (1944); these Rev. 7, 149], now applying them to the problem of the conformal representation of arbitrary domains on canonical domains. The method yields numerical approximations through the solutions of systems of linear equations.

E. F. Beckenbach (Princeton, N. J.).

Theory of Series

Hornich, Hans. Su alcune successioni di serie i cui termini generali convergono a zero. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 10, 298–300 (1951).

Let p be an integer greater than 2, and let Z be the set of pth roots of unity. The author [Monatsh. Math. Phys. 45, 432–434 (1937)] showed that Z is a sum-factor set in the following sense. If A is a complex number and $\sum a_n$ is a series of complex numbers for which $\sum|a_n|=\infty$ and $a_n\to 0$, then there is a sequence z_n of numbers in Z such that the series $\sum a_n z_n$ converges to A. Calabi and Dvoretzky [Trans. Amer. Math. Soc. 70, 177–194 (1951); these Rev. 12, 604] generalized this theorem of Hornich by characterizing sum-factor sets. The paper under review gives another generalization involving a countable set of given double series. Let $A_1, A_2, \cdots$ be a sequence of complex numbers. For each $j=1, 2, 3, \cdots$, let $\sum a_{nk}(j)$ be a double series such that $\sum_{n\neq j}\sum_{k=1}^{\infty}|a_{nk}(j)|<\infty$, $\sum_{k=1}^{\infty}|a_{jk}(j)|=\infty$, and $a_{jk}(j)\to 0$ as $k\to\infty$. Then there is a double sequence z_{nk} of numbers in Z such that $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}a_{nk}(j)z_{nk}=A_j$ for each $j=1, 2, \cdots$. [In this review, the new and convenient terminology and notation of Calabi and Dvoretzky are used; the author took ζ to be a primitive pth root of unity and represented z_{nk} in the form $\zeta^{l_{nk}}$.]

R. P. Agnew (Ithaca, N. Y.).

Rechard, O. W. A note on the summability of infinite series by sequence to sequence and series to sequence transformations. Proc. Amer. Math. Soc. 2, 730–731 (1951).

The author observes that corresponding to each regular (matrix) sequence-to-sequence transformation A it is possible to construct a regular series-to-sequence transformation B, but that the converse is generally false. This apparent advantage of B over A vanishes under the condition indicated in the author's theorem, namely, if $B=(b_{ij})$ is a regular series-to-sequence transformation which evaluates some divergent series with bounded partial sums, and if $a_{ij}=b_{ij}-b_{i,j+1}$, then the sequence-to-sequence transforma-

tion $A=(a_{ij})$ is regular, and a divergent series with bounded partial sums is evaluated by B to T if, and only if, it is evaluated by A to T. *J. D. Hill* (East Lansing, Mich.).

Hill, J. D. The Borel property of summability methods. Pacific J. Math. 1, 399–409 (1951).

A method of summation $T=(a_{ik})$ has the Borel property if for almost all $\alpha=0.\alpha_1\alpha_2\cdots$ the sequence α_n of the digits of the dyadic fraction of α is T-summable to $\frac{1}{2}$. The author gives necessary conditions and sufficient conditions for a method T to have this property, mostly in terms of the $A_n=\sum_k a_{nk}^2$. (1) The conditions $A_n<+\infty$ for each n and $A_n=o(1)$ are necessary. (2) Conditions (1) are not sufficient, even if the method T is regular (this result is attributed to Erdős). (3) $\sum_n \exp(-\delta A_n^{-1})<+\infty$ for each $\delta>0$ is sufficient. (4) No condition of the form $A_n\psi(n)=o(1)$, $\psi(n)\to\infty$ is necessary. The proofs are based on some properties of Rademacher's functions due to Kolmogoroff and Khinchine. *G. G. Lorentz* (Toronto, Ont.).

Jackson, F. H. Proper extensions by dilution of matrices. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 308–314 (1951).

Let $T=(a_{n,k})$ be a Toeplitz matrix, and let $T^{(m)}$ be the related Toeplitz matrix, with $a^{(m)}_{n,mk}=a_{n,k}$ and with $a_{n,t}=0$ when t is not divisible by m; that is, let $T^{(m)}$ be obtained by an m-fold dilution of T with columns of zeros. If the matrix T satisfies certain conditions, and the arguments of the singular points of the function $f(z)=\sum a_n z^n$ are suitably restricted, then the domain in which $T^{(m)}$ sums to $f(z)$ the sequence $\{s_k(z)\}$ associated with $\sum a_n z^n$ contains the corresponding domain for T as a proper subset. The author applies his result to matrices of Mittag-Leffler type and to Euler-Knopp matrices. *G. Piranian.*

Jurkat, Wolfgang. Über Konvergenzfaktoren bei Rieszschen Mitteln. Math. Z. 54, 262–271 (1951).

A well known theorem of Bromwich and Kojima, for $(C,1)$-summability, is the case $\lambda_n=n$ of the following result: necessary and sufficient conditions for $\sum a_n b_n$ to be summable $(R,\lambda_n,1)$ whenever $\sum a_n$ is summable $(R,\lambda_n,1)$ are that (i) $b_n=O\{(\lambda_{n+1}-\lambda_n)/\lambda_{n+1}\}$, (ii) $\sum\lambda_{\nu+1}|\Delta(\Delta b_\nu/\Delta\lambda_\nu)|<\infty$ [T. J. I'A. Bromwich, Math. Ann. 65, 350–369 (1908); T. Kojima, Tôhoku Math. J. 12, 291–326 (1917); for further references see G. H. Hardy, Divergent Series, Oxford, 1949, p. 146; these Rev. 11, 25]. The author gives the result for general λ_n ($\lambda_n\uparrow\infty$), with the superfluous condition $\Delta b_n/\Delta\lambda_n=o(1)$. [But (ii) implies $\sum|\Delta(\Delta b_\nu/\Delta\lambda_\nu)|<\infty$, and hence there is a number s such that

$$|(\Delta b_n/\Delta\lambda_n)-s|\leqq\sum_n^\infty|\Delta(\Delta b_\nu/\Delta\lambda_\nu)|=o(1),$$

so that $b_n=s\lambda_n+o(\lambda_n)$. This is only consistent with (i) if $s=0$.] Some related problems are discussed under various restrictions. *L. S. Bosanquet* (London).

Borwein, D. On the absolute Cesàro summability of integrals. Proc. London Math. Soc. (3) 1, 308–326 (1951).

Let $\lambda\geqq 0$, λ being not necessarily an integer. The integral $\int_1^\infty f(t)dt$ is evaluable to I by the Cesàro method (C,λ) if the transform $F_\lambda(t)=t^{-\lambda}\int_1^t(t-u)^\lambda f(u)du$ exists as a Lebesgue integral for each $t\geqq 0$ and $F_\lambda(t)\to I$ as $t\to\infty$. The integral is absolutely evaluable (C,λ), or is evaluable $|C,\lambda|$, if $F_\lambda(t)$ has bounded variation over the infinite interval $t\geqq 1$. The author gives three conditions on a function $\phi(t)$ which are sufficient to ensure that $\int_0^\infty f(t)\phi(t)dt$ is evaluable $|C,\lambda|$ whenever $\int_0^\infty f(t)dt$ is evaluable $|C,\lambda|$. Two of these conditions are shown to be best possible when the other one holds. The conditions are analogous to those obtained by Hardy [Messenger of Math. 40, 87–91 (1911)] and Cossar [J. London Math. Soc. 16, 56–68 (1941); these Rev. 3, 109] for (C,r) evaluability of integrals of products and by Fekete [Math. és Termész. Ert. 35, 309–324 (1917)] and Bosanquet [J. London Math. Soc. 20, 39–48 (1945); these Rev. 7, 432] for $|C,r|$ evaluability of series of products. *R. P. Agnew* (Ithaca, N. Y.).

Hsu, L. C. The asymptotic behaviour of a kind of multiple integrals involving a parameter. Quart. J. Math., Oxford Ser. (2) 2, 175–188 (1951).

The author generalizes Laplace's classical theorem on the asymptotic expression of an integral for an n-fold integral of the form $\int\cdots\int F(x_1,\cdots,x_n;\lambda)dx_1\cdots dx_n$. The integrand is a positive, real-valued function, which attains an absolute maximum at an interior point $\xi(\lambda)$. The author gives sufficient conditions for which

$$\int\cdots\int\exp[f(x;\lambda)]dx_1\cdots dx_n\sim\exp[f(\xi;\lambda)]\left(\frac{(2\pi)^n}{H_n[-f(\xi;\lambda)]}\right)^{\frac{1}{2}}.$$

Here $H_n[-f]$ denotes the Hessian of $-f$. The case in which $f(x;\lambda)=\lambda g(x)$ and the function $g(x)$ takes a maximum value at a boundary point has been discussed in a previous paper by the same author [Duke Math. J. 15, 623–632 (1948); these Rev. 10, 246]. *H. A. Lauwerier* (Amsterdam).

Fourier Series and Generalizations, Integral Transforms

Chow, Hung Ching. Theorems on power series and Fourier series. Proc. London Math. Soc. (3) 1, 206–216 (1951).

If $f(z)$ is a power series of class H^p, $1<p\leqq 2$, and $s_n^\alpha(\theta)$ is the (C,α) mean of the series at $e^{i\theta}$, then

$$\sum_{\nu=0}^{n}|s_\nu^\alpha(\theta)-f(e^{i\theta})|^p=O(n)$$

for $\alpha>(1/p)-1$ and $\sum_{\nu=0}^{n}\nu^{-1}|s_\nu^\alpha(\theta)-f(e^{i\theta})|^p=O(\log n)$ for $\alpha=(1/p)-1$. For the same class of functions $\{\log(n+1)\}^{1+\delta}$, $\delta>0$, is a summability factor $|C,1/p|$ at $e^{i\theta}$. Suppose $u(\theta)$ is Lebesgue integrable, $0\leqq\theta\leqq 2\pi$, s_n^α is the (C,α) mean of the Fourier series of $u(\theta)$ and $\int_0^\pi|\varphi(\theta_0,t)|^p t^{-1}dt<\infty$ where

$$\varphi(\theta_0,t)=\tfrac{1}{2}\{u(\theta_0+t)+u(\theta_0-t)-2u(\theta_0)\}.$$

Then $\sum|\sigma_n^\alpha(\theta_0)-u(\theta_0)|^p n^{-1}<\infty$ for $\alpha=(1/p)-1$ and $\{\log(n+1)\}^{-(p-1)/(p-\delta)}$, $\delta>0$, is a summability factor $|C,1/p|$ at θ_0. *P. Civin* (Eugene, Ore.).

Davydov, N. A. Generalization of some theorems on the convergence of power and trigonometric series. Doklady Akad. Nauk SSSR (N.S.) 80, 317–320 (1951). (Russian)

The author discusses the behavior of lacunary trigonometric and power series, and the behavior of nonlacunary series when the arguments (but not the moduli) of the coefficients are changed. The following are some of the results obtained. 1) In any power series $a_0+a_1e^{i\theta}+a_2e^{2i\theta}+\cdots$ satisfying $a_k=o(1)$, $\sum|a_k|^2=\infty$, the arguments of the a_k

may be so changed that for almost every θ the partial sums of the resulting series are dense in the complex plane. 2) If the lacunary series (*) $\sum_1^\infty a_k z^{\lambda_k}$ with $\lambda_{k+1}/\lambda_k > 3$ for all k converges for $|z|<1$, then no matter how we change the arguments of the a_k, the sum (*) is unbounded in the neighborhood of every point on $|z|=1$. 3) If $a_k = O(1)$, $\lambda_{k+1}/\lambda_k > \lambda > 1$, then a necessary and sufficient condition for convergence of (*) at a point z_0, $|z_0|=1$, is the existence of the limit of the function (*) along some non-tangential path ending at z_0. [A result more general than 2) can be found in the reviewer's note, Studia Math. 3, 77–91 (1931).] *A. Zygmund.*

Stečkin, S. B. On de la Vallée Poussin sums. Doklady Akad. Nauk SSSR (N.S.) 80, 545–548 (1951). (Russian)

The de la Vallée Poussin sums of a Fourier series with partial sums $s_n(x)$ are

$$\sigma_{n,m}(x) = (m+1)^{-1} \sum_{k=n-m}^{n} s_k(x).$$

The author investigates in more detail than has previously been done the Lebesgue constants $N_{n,m}$ for these sums, $N_{n,m} = \sup \|\sigma_{n,m}(x, f)\|$ for $f(x)$ continuous with $\|f\| \leqq 1$, the norm being the maximum. Among the results are the following. If $n_p \geqq m_p \to \infty$ (even integers) and

$$r_p = 2(n_p+1)/(m_p+1) - 1 \to r < \infty$$

then

$$N_{n_p, m_p} \to 2\pi^{-1} \int_0^\infty |\sin rt \cdot \sin t| t^{-2} dt.$$

As a function of $(n+1)/(m+1)$, $N_{n,m}$ is increasing and concave. There is the following asymptotic formula generalizing that for the ordinary Lebesgue constants ($m=0$):

$$N_{n,m} = \frac{4}{\pi^2}\left\{\log\frac{n+1}{m+1} + 2\sum_{\nu=1}^{\infty}\frac{\log \nu}{4\nu^2-1} + 3\log 2 + \gamma\right\} + O\left(\frac{m+1}{n+1}\right).$$

R. P. Boas, Jr. (Evanston, Ill.).

Boas, R. P., Jr. Partial sums of Fourier series. Proc. Nat. Acad. Sci. U. S. A. 37, 414–417 (1951).

Résultats très interessants sur les changements de signe dans la suite: $s_n(0) - f(0) = l_n(0)$; $f(n)$ designant une fonction sommable paire de période 2π, $s_n(x)$ la somme partielle d'ordre n de sa série de Fourier, $f(x)$ satisfaisant de plus à des conditions variées. L'auteur établit, par exemple, que: (i) si $f(x)$ est constante au voisinage de zéro, $l_n(0)$ et $l_{n+1}(0)$ ne peuvent avoir le même signe pour une suite de valeurs de densité 1; (ii) si $f(x)$ a une dérivée continue d'ordre p pour $|x| \leqq \delta$ ($\delta > 0$) et si $l_n(0)$ ne change pas de signe à partir d'un certain rang, alors $a_n = O(n^{-p})$ (a_n: coefficiente de Fourier d'ordre n de f); autrement dit, une part importante de la régularité admise pour f peut se prolonger.

J. Favard (Paris).

Boas, R. P., Jr. Completeness of sets of translated cosines. Pacific J. Math. 1, 321–328 (1951).

Theorem. Let $\{\lambda_n\}_0^\infty$ be an increasing, unbounded sequence of non-negative numbers, $N_1(r)$ the number of λ_{2n} not exceeding r, $N_2(r)$ the number of λ_{2n+1} not exceeding r. If

$$\int_1^r t^{-1} N_1(t)\, dt > \tfrac{1}{2}r - c\log r - \text{const.}$$

and

$$\int_1^r t^{-1} N_2(t)\, dt > \tfrac{1}{2}r - (c+\tfrac{1}{2})\log r - \text{const.},$$

where $c = (p-1)/2p$ ($1 \leqq p < \infty$), $c < \frac{1}{2}$ ($p = \infty$), then

$$\cos\lambda_{2n}t + a_{2n}\sin\lambda_{2n}t, \quad -a_{2n+1}\cos\lambda_{2n+1}t + \sin\lambda_{2n+1}t$$

is L^p-complete on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, if the a_n are real numbers, all of the same sign. Specializations of the theorem give results like: $\{\sin(nx+q_n)\}_0^\infty$, is L-complete on $(0, \pi)$, if $|n+1-\lambda_n| \leqq \delta < \frac{1}{4}$, $\frac{1}{4}\pi\delta \leqq q_n < \frac{1}{4}\pi(1-\delta)$. The proof of the theorem is based on a complex variable investigation of $H(z) = \int_{-\pi/2}^{\pi/2} f(t)\cos zt\, dt \int_{-\pi/2}^{\pi/2} f(u)\sin zu\, du$.

W. H. J. Fuchs (Ithaca, N. Y.).

Hsu, L. C. The representation of functions of bounded variation by singular integrals. Duke Math. J. 18, 837–844 (1951).

The author derives conditions under which the equation

$$(*) \qquad \lim_{\lambda\to\infty}\int_E f(t)\phi(t-x,\lambda)\,dt = \tfrac{1}{2}[f(x+0)+f(x-0)]$$

holds. Here E is a measurable set of real numbers, x is a point of density of E, f is of bounded variation on E (in the sense of Saks, Theory of the Integral [Warsaw-Lvov, 1937], p. 221) and $\phi(t, \lambda)$ is a kernel with certain properties. One difference between this paper and earlier work on singular integrals is that E need not be an interval. The point x is called a point of increasing density of E if $\rho(x, t) \to 1$ monotonically as $t \to x$, where

$$\rho(x,t) = \frac{1}{t-x}\int_x^t X(u)\,du,$$

and X is the characteristic function of E. Theorem 1 states that (*) holds if x is a point of increasing density of E, and if ϕ satisfies a certain set of five conditions (which are, for instance, satisfied by the Dirichlet kernel). Theorem 2 deals with a more specialized situation, in which the kernel is of the form $\phi(\lambda t)/t$. Some of Lebesgue's results on singular integrals [Ann. Fac. Sci. Univ. Toulouse (3) 1, 25–117, 119–128 (1909)] are used in the proofs. *W. Rudin.*

Arnol'd, G. A. On the order of approximation of continuous functions by singular integrals of a certain special form. Mat. Sbornik N.S. 29(71), 427–432 (1951). (Russian)

The author shows that if one sets

$$P_n(f(x), x) = \int_a^b \phi(x,t)^n f(t)\,dt / K_n,$$

where $K_n = \int_a^b \phi(x,t)^n dt$, then under certain simple assumptions concerning ϕ and its partial derivatives with respect to x and t, one has $\lim_{n\to\infty} n(P_n - f(x)) = L_0(x)f'(x) + L_1(x)f''(x)$ where L_0 and L_1 are explicit functions of ϕ and its partial derivatives. Some examples are given. *R. Bellman.*

Campbell, Robert. Sur les sommations de Césaro d'ordre entier des séries de Weber. C. R. Acad. Sci. Paris 233, 596–598 (1951).

A necessary and sufficient condition for the Cesàro summability of order one, to the sum s, of the Hermite series of $f(x)$ is

$$\lim_{n\to\infty}\int_0^\infty [f(x+t)+f(x-t)-2s]K_n^1(x,t)\,dt = 0$$

where K_n^1 is the kernel. The Gibbs phenomenon is present in this summation, but disappears if Cesàro summation is applied to the sub-sequence $\{S_{n^2}\}$ of the sequence $\{S_n\}$ of partial sums. *A. Erdélyi* (Pasadena, Calif.).

* **Jessen, Børge. Mean motions and almost periodic functions.** Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 76–92. University of Toronto Press, Toronto, 1951. $6.00.

Exposé des résultats obtenus sur le problème des moyens mouvements relatif aux fonctions presque-périodiques (c'est-à-dire sur le comportement asymptotique de l'argument d'une telle fonction) et divers problèmes annexes, problèmes qui ont été étudiés avec beaucoup de soins et de détails par divers auteurs. Les difficultés rencontrées, les méthodes employées pour les surmonter sont exposées avec clarté et l'enchaînement des idées est facile à suivre. *J. Favard.*

Wing, G. M. On the L^p theory of Hankel transforms. Pacific J. Math. 1, 313–319 (1951).

E. Hille and J. D. Tamarkin [Bull. Amer. Math. Soc. 39, 768–774 (1933)] have proved: Let $1 \leqq p \leqq 2$, $f(x) \in L_p(-\infty, \infty)$ and $g(t)$ be the Fourier transform of $f(x)$; (A) for $1 < p \leqq 2$ the function $f_a(x) = \int_{-a}^{a} e^{ixt} g(t)\,dt$ converges to $f(x)$ in the mean of order p as $a \to \infty$; (B) this does not hold for $p = 1$ [for (B), cf. Kober, Quart. J. Math. Oxford Ser. 12, 78–85 (1941); these Rev. 3, 38]. The author extends both results to the Hankel transform $g(t)$ of $f(x) \in L_p(0, \infty)$. The proof of (A) was based on the relations

$$f_a(x) = \int_{-\infty}^{\infty} \kappa(x, u, a) f(u)\,du; \quad \int_{-\infty}^{\infty} |f_a(x) - f(x)|^p dx \to 0 \qquad (1 < p \leqq 2;\ a \to \infty)$$

where κ is the Dirichlet kernel $\pi^{-1}(x-u)^{-1} \sin a(x-u)$. The author proceeds in a similar way, replacing κ by the, in substance more general, kernel

$$a(xu)^{\frac{1}{2}}\{uJ_{\nu+1}(ua)J_\nu(xa) - xJ_{\nu+1}(xa)J_\nu(ua)\}(u^2-x^2)^{-1}$$

where $\Re(\nu) \geqq -\frac{1}{2}$. Finally he deals with the crucial case $p = 1$, i.e. $f(x) \in L(0, \infty)$. He uses a summability method. Taking $k > 0$ and $\kappa(x, u, a) = \int_0^a (xu)^{\frac{1}{2}} J_\nu(xt) J_\nu(ut)(1 - t/a)^k dt$ he shows that $\int_0^a |f_a(x) - f(x)|\,dx \to 0$ $(a \to \infty)$ and, therefore, the function $\int_0^a (xu)^{\frac{1}{2}}(1 - u/a)^k J_\nu(xu) g(u)\,du$ converges strongly to $f(x)$. *H. Kober* (Birmingham).

Hirschman, I. I., Jr., and Widder, D. V. Convolution transforms with complex kernels. Pacific J. Math. 1, 211–225 (1951).

Cet article généralise des articles antérieurs. Les formules de l'article, "The inversion of a general class of convolution transforms" [Trans. Amer. Math. Soc. 66, 135–201 (1949); ces Rev. 11, 350], sont encore valables pour une fonction entière $E(s)$ à zéros complexes, de la forme $E(s) = \prod_{k=1}^{\infty}(1 - (s/a_k)) \exp(s/b_k)$, $a_k = b_k + ic_k$, pourvu que $\sum_{k=1}^{\infty}(1/b_k^2) < \infty$, $\sum_{k=1}^{\infty}(c_k/b_k)^2 < \infty$. Les formules de l'article, "Generalized inversion formulas for convolution transforms, II" [Duke Math. J. 17, 391–402 (1950); ces Rev. 12, 497], sont encore valables pour $E(s) = \prod_{k=1}^{\infty}(1 - (s^2/a_k^2))$, si $b_k > 0$, $b_k \sim \Omega k/\pi$, $\sum_{k=1}^{\infty}(c_k/b_k)^2 < \infty$. *L. Schwartz.*

Delange, Hubert. Remarque sur une formule d'inversion de l'intégrale de Laplace-Stieltjes. Bull. Sci. Math. (2) 75, 146–152 (1951).

Let $\alpha(t)$, defined for $0 \leqq t < \infty$, be of bounded variation on every finite interval, and let

$$\alpha(0) = 0, \quad \alpha(t) = \tfrac{1}{2}[\alpha(t+) + \alpha(t-)].$$

If $f(s) = \int_0^\infty e^{-st}\,d\alpha(t)$, then

$$\int_0^t |d\alpha(u)| = \lim_{x \to +\infty} \sum_{q=0}^{E[xt]} \frac{x^q}{q!} |f^{(q)}(x)|$$

where $E[xt]$ designates the largest integer not exceeding xt. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

Charles, Henri. Sur l'inversion de certaines images irrationnelles. Bull. Soc. Roy. Sci. Liège 20, 327–330 (1951).

Let $f(s)$ denote the Laplace transform of $F(t)$, let $p = [s^{\frac{1}{2}} + (s+\lambda)^{\frac{1}{2}}]^2$ and $q = s + (s^2+\lambda^2)^{\frac{1}{2}}$ where λ is a constant. Formulas are derived for the inverse transforms of $f(p)[s(s+\lambda)]^{-\frac{1}{2}}$ and $f(q)(s^2+\lambda^2)^{-\frac{1}{2}}$ in terms of integrals involving $F(t)$ and Bessel functions. The inverse transforms of two exponential functions are found with the aid of those formulas. *R. V. Churchill* (Ann Arbor, Mich.).

Doetsch, Gustav. Über die endliche Laplace-Transformation. Math. Ann. 123, 411–414 (1951).

Let $f(s) = \int_0^\infty e^{-st} F(t)\,dt$. The author discusses conditions for $F(t)$ to vanish for $t \geqq h > 0$. *I. I. Hirschman, Jr.*

Doetsch, G. On the problem of convergence in the theory of the Laplace transform. Revista Unión Mat. Argentina 15, 19–23 (1951). (Spanish)

Expository article. *I. I. Hirschman, Jr.*

Polynomials, Polynomial Approximations

Videnskiĭ, V. S. On estimates of derivatives of a polynomial. Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 401–420 (1951). (Russian)

The principal content of this paper consists of detailed proofs of results previously announced [Doklady Akad. Nauk SSSR (N.S.) 67, 777–780 (1949); 73, 257–259 (1950); these Rev. 11, 104; 12, 95]. *R. P. Boas, Jr.*

Karapandjitch, Georges. Une contribution à l'étude des zéros des polynômes. Bull. Soc. Math. Phys. Serbie 2, nos. 3–4, 43–46 (1950). (Serbo-Croatian. French summary)

The substitutions

$$p = m(n-1), \quad A = -1/n(y')^m, \quad B = (y')^{1-m}x, \quad C = \alpha/(y')^m,$$

change the algebraic equation (1) $y^p + Ay^n + By^{n-1} + C = 0$ into a differential equation

$$(2) \qquad y^{m(n-1)}(y')^m - y^n/n + y'xy^{n-1} + \alpha = 0,$$

with known solution (3) $n(k^m + kx + \alpha) = y^n$, for which (4) $y' = k/(n(k^m + kx + \alpha))^{(n-1)/n}$. The algebraic equation (1) can now be solved using the differential equation (2), by noting that p, n, A, B, and C in (1) determine m, y', x, and α in (2). If k is obtained from (4) a root of (1) is then given by (3). [It seems to the reviewer that the determination of k in (4) may be as difficult as finding a root of (1) by the usual methods.] If k is preassigned, then (4) yields an equation in A, B, C, m, and n which must be satisfied for (3) to give a root of (1). This is illustrated with an example polynomial for the case $k = 1$.

The author remarks that the use of differential equations for solving algebraic equations has not been exploited, and that the above is merely an example illustrating the possibilities in that direction. *A. W. Goodman.*

Froda, Alexandru. Existence of intervals of contraction in a class of polynomials of degree n. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 461–465 (1950). (Romanian. Russian and French summaries)

This note is concerned with a proof of the well-known theorem: For a real polynomial $P_n(x)$ of degree n such that $P_n(a)=P_n(b)=0$, there exists an interval (a', b') interior to (a, b) and independent of the coefficients of $P_n(x)$ such that (a', b') always contains a zero of the derivative $P_n'(x)$ of the polynomial. The interval (a', b') is called the interval of contraction. *E. Frank* (Chicago, Ill.).

Mamuzitch, Zlatko. Sur le théorème I de Weierstrass sur l'approximation par polynomes des fonctions continues d'une variable réelle. Bull. Soc. Math. Phys. Serbie **2**, nos. 3–4, 61–72 (1950). (Serbo-Croatian. French summary)

Essentially an historico-expository paper on the theorem of the title, with some observations relating to the Lebesgue proof of that theorem. *I. M. Sheffer.*

Levi, Eugenio. Ancora sopra un'applicazione dei polinomi di Bernstein all'approssimazione in media delle funzioni sommabili. Atti Accad. Naz. Lincei. Rend. Cl Sci. Fis. Mat. Nat. (8) **10**, 360–364 (1951).

The author shows that if $f(x, y)$ belongs to L^p, $p\geqq 1$, for $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, and if $F(x, y)=\int_0^y\int_0^x f(s, t)\,ds\,dt$, then the second-order mixed partial derivatives of the Bernstein polynomials of $F(x, y)$ converge in mean of order p to $f(x, y)$. The method is the same as that of a previous note [same Atti Cl. Sci. Fis. Mat. Nat. (8) **9**, 242–246 (1950); these Rev. **12**, 701] for one variable and $p=1$. *R. P. Boas, Jr.*

Berman, D. L. Interpolation in a many-dimensional space. Doklady Akad. Nauk SSSR (N.S.) **81**, 9–12 (1951). (Russian)

It is well known that the Lagrange interpolation polynomials associated with a given system of nodes and a given continuous function do not necessarily converge to the function. However, S. Bernstein [Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **5**, 49–57 (1932)] has given interpolation processes which do converge, and his results have been generalized by the author [same Doklady (N.S.) **58**, 1569–1571 (1947); **60**, 333–336 (1948); these Rev. **9**, 348, 584]. He now extends these results to interpolation of functions of several variables. [The interpolation process of L. Merli, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7** (1949), 212–216 (1950); these Rev. **11**, 662, is a special case of Bernstein's.] *R. P. Boas, Jr.* (Evanston, Ill.).

Wolibner, W. Sur un polynôme d'interpolation. Colloquium Math. **2**, 136–137 (1951).

Given n points (a_k, b_k) $(k=1, \cdots, n)$ in the plane such that $a_{k-1}<a_k$ and $b_{k-1}\neq b_k$, let $\varphi(x)$ be the function whose graph consists of line segments joining the successive points. The author constructs a polynomial $p(x)$ such that $p(a_k)=b_k$ $(k=1, \cdots, n)$ and $p(x)$ increases or decreases simultaneously with $\varphi(x)$ in the interval $a_1\leqq x\leqq a_n$. *W. F. Eberlein* (Madison, Wis.).

Călugăreanu, George. On Čebyšev polynomials of bounded closed plane sets. II. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 7–15 (1950). (Romanian. Russian and French summaries)

Let E be the set formed by the N points $z=c_1, \cdots, c_N$ of the complex plane, and let $T_n(z; E)$ be the Čebyšev polynomial associated with this set. For $n<N$, the author proves that the polynomial of exact degree n and leading coefficient unity which minimizes

$$\left\{\frac{1}{N}\sum_{\nu=1}^{N}|P(c_\nu)|^{2p}\right\}^{1/2p},$$

p a positive integer, tends to $T_n(z, E)$ as $p\to\infty$. The minimum itself tends to $\max T_n(z, E)$, $z\in E$. *A. Erdélyi.*

Special Functions

Tricomi, F. G., and Erdélyi, A. The asymptotic expansion of a ratio of gamma functions. Pacific J. Math. **1**, 133–142 (1951).

The following asymptotic expansion is proved

$$\frac{\Gamma(z+d)}{\Gamma(z+\beta)}\sim\sum_{n=0}^{\infty}\frac{a_n z^{\alpha-\beta-n}}{\Gamma(\alpha-\beta-n+1)}\quad\text{as } z\to\infty$$

valid for all $\alpha, \beta, \alpha-\beta\neq -1, -2, \cdots$ and the complex z-plane slit from $-\alpha$ to $-\alpha-\infty$. The coefficients a_n can be derived from a given generating function and can also be expressed in terms of generalized Bernoulli polynomials. Moreover, a recurrent relation is derived between the first $n+1$ coefficients. The simple case $\beta=0$ is treated separately. The remainder, however, is not discussed.

H. A. Lauwerier (Amsterdam).

Tricomi, Francesco G. Expansion of the hypergeometric function in series of confluent ones and application to the Jacobi polynomials. Comment. Math. Helv. **25**, 196–204 (1951).

In earlier papers [Giorn. Ist. Ital. Attuari **12**, 14–33 (1941); Ann. Mat. Pura Appl. (4) **26**, 141–175 (1947); **28**, 263–289 (1949); these Rev. **8**, 268; **10**, 605; **12**, 96] the author has given expansions of confluent hypergeometric functions in series of Bessel functions. He now applies the Laplace transformation to one of these expansions and obtains infinite series of confluent hypergeometric functions (or Bessel functions) for (1) $F(a, b; c; 2z/(2k+z))$. He notes that the order of magnitude of successive terms decreases rapidly if $|k/z|$ is large, deduces certain results on "small" zeros of (1), and applies his results to Jacobi and Legendre polynomials. *A. Erdélyi* (Pasadena, Calif.).

Chandra, Dinesh. On the Hankel transformation of generalised hypergeometric functions. Bull. Calcutta Math. Soc. **43**, 13–16 (1951).

The author computes the Hankel transform of order ν of a function of type ${}_3F_3$ and remarks that Hari Shanker's result [J. London Math. Soc. **21**, 194–198 (1946); these Rev. **8**, 460] and other known results are special cases of his formula. *A. Erdélyi* (Pasadena, Calif.).

Merli, Luigi. Su alcune disuguaglianze riguardanti i polinomi ultrasferici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 371–374 (1951).

Let $\{P_n^\lambda(x)\}$ be the ultraspherical polynomials, and for given λ, set

$$F_n(x)=P_n^\lambda(x)/P_n^\lambda(1),\quad \Delta_n^\lambda(x)=[F_n(x)]^2-F_{n-1}(x)F_{n+1}(x).$$

Following a method of Sansone [Boll. Un. Mat. Ital. (3) **4**, 221–223 (1949); these Rev. **11**, 432] a first order non-homogeneous linear differential equation is obtained for

$\Delta_n^\lambda(x)$, and this is used to establish the following inequalities, analogous to results of Szász [Proc. Amer. Math. Soc. 1, 256–267 (1950); these Rev. 11, 662]:

$$\frac{\lambda[1-F_n^2(x)]}{n(n+2\lambda)} \leqq \Delta_n^\lambda(x) \leqq \Delta_n^\lambda(0) + \frac{\lambda}{n(n+2\lambda)}[F_n^2(0)-F_n^2(x)]$$

for $0<\lambda<1$, $-1\leqq x\leqq 1$;

and

$$\max\left\{0, \Delta_n^\lambda(0)+\frac{\lambda[F_n^2(0)-F_n^2(x)]}{n(n+2\lambda)}\right\} \leqq \Delta_n^\lambda(x) \leqq \frac{\lambda[1-F_n^2(x)]}{n(n+2\lambda)}$$

for $\lambda>1$, $-1\leqq x\leqq 1$.

It is also shown that $\Delta_n^\lambda(x)<0$ for $\lambda>0$, $|x|>1$.

I. M. Sheffer (State College, Pa.).

Tomić, M. Zur Theorie der Legendreschen Polynome. Srpska Akad. Nauka. Zbornik Radova, Knj. 7. Matematički Institut, Knj. 1, 73–82 (1951). (Serbo-Croatian. German summary)

The author discusses Bernstein's inequality

$$|P_n(\cos\theta)-P_{n+1}(\cos\theta)| \leqq A(\theta/n)^{\frac{1}{2}}$$

for Legendre polynomials. For a general class of polynomials

$$P_n(\cos\theta)=\sum_{m=0}^{n} c_m c_{n-m}\cos(n-2m)\theta, \quad c_m>c_{m+1},$$

he estimates in a similar fashion

$$P_n(\cos\theta)-P_n(1)-[P_{n+1}(\cos\theta)-P_{n+1}(1)].$$

[In the paper, the value of $P(\cos\theta)$ for $\theta=0$ is denoted by $P(0)$.] He adds a few remarks on points connected with the asymptotic representation of $P_n(\cos\theta)$ and the conjugate polynomials. *A. Erdélyi* (Pasadena, Calif.).

Forsythe, George E. Second order determinants of Legendre polynomials. Duke Math. J. 18, 361–371 (1951).

Denoting by $P_r(x)$ the classical Legendre polynomial of degree r, the function of the real variable x,

$$\Delta=\Delta(n,h,k;x)=P_n(x)P_{n+h+k}(x)-P_{n+h}(x)P_{n+k}(x),$$

is considered for any triple of integers (n,h,k) such that $n\geqq 0$, $k\geqq h\geqq 1$. The determinant Δ is said to have property T if $\Delta(n,h,k;x)<0$ in $0<x<1$. Turán discovered that $\Delta(n,1,1;x)$ has property T for all $n\geqq 0$. The author investigates for what triples (n,h,k) the determinant Δ has property T. It is shown that for any $n\geqq 0$, Δ has property T for the triples $(n,1,2)$, $(2n+1,2,2)$. On the other hand, the author proves that property T fails for a set of triples (n,h,k) which has an asymptotic density of 7/8 within the class of all triples. No conclusion is reached for the remaining set. *W. Seidel* (Rochester, N. Y.).

Mitra, S. C., and Sharma, A. On generating functions of polynomials. I. Generalised parabolic cylinder functions of Weber. Bull. Calcutta Math. Soc. 43, 46–50 (1951).

"Generalised parabolic cylinder functions" are, on the authors' own showing, disguised Laguerre polynomials, and the generating functions obtained in this paper are disguised forms of the well-known generating functions of Laguerre polynomials. *A. Erdélyi* (Pasadena, Calif.).

Sips, Robert. Recherches sur les fonctions paraboliques. Bull. Soc. Roy. Sci. Liège 20, 355–373 (1951).

The partial differential equation $u_{xx}+u_{yy}-u=0$ has the solutions $I_0(R)$, $K_0(R)$ where $R^2=x^2+y^2$. Transformation to parabolic coordinates leads to solutions in terms of parabolic cylinder functions, and the relation between these two types of solutions leads to numerous formulas. *A. Erdélyi.*

Sips, Robert. Détermination directe de l'exposant caractéristique de l'équation de Hill. Bull. Soc. Roy. Sci. Liège 19, 406–416 (1950).

The author considers the equation of Mathieu and its solutions corresponding to a general value of the characteristic exponent, while the values of the independent variable lie between 0 and 2π. In this interval these general solutions are developed in a series of Mathieu functions, that is of periodic solutions of the Mathieu equation in this interval. As special cases the general solutions are identified with the so called second solutions of the Mathieu equation correlated with the Mathieu functions indicated above. Using series for these Mathieu functions these second solutions are evaluated in some numerical cases. *M. J. O. Strutt.*

Meixner, Josef. Integralbeziehungen zwischen Mathieuschen Funktionen. Math. Nachr. 5, 371–378 (1951).

The author considers the Mathieu functions and the corresponding second solutions of the Mathieu equation. As a different characterization of the solutions of Mathieu's equation they are defined by their asymptotic character. The integral relations arrived at in the paper are derived from the known developments of general Mathieu functions in exponential series. Besides these relations having fixed bounds, further relations with variable bounds are derived from an integral of this type involving a Bessel function and a Mathieu function. Then products of two Mathieu functions are considered. Expressions for integrals of such products are derived. *M. J. O. Strutt* (Zurich).

Pritchard, R. L. Discussion of papers by Pachner and by Stenzel on radiation from a circular emitter. J. Acoust. Soc. Amer. 23, 591 (1951).

The author gives a proof of

$$\int_0^\infty \frac{J_\nu[x(t^2+k^2)^{\frac{1}{2}}]}{(t^2+k^2)^{\nu/2}}\frac{J_\mu[y(t^2+k^2)^{\frac{1}{2}}]}{(t^2+k^2)^{\mu/2}}dt = \frac{(-1)^\nu}{xk^{\mu+\nu}}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{1}{2})}{m!\Gamma(\frac{1}{2})}\left(\frac{y}{x}\right)^m J_{\mu+m}(ky)C_{m-\nu}(kx),$$

where ν, μ are real, $\nu+\mu>0$, $y/x<1$,

$$C_n(z)=(-1)^n(\pi z/2)^{\frac{1}{2}}J_{-n-\frac{1}{2}}(z).$$

This extends a result due to Stenzel [Ann. Physik (5) 41, 245–260 (1942); these Rev. 5, 137] for $\nu=0$, $\mu=1$. The derivation of Stenzel was not quite correct, as has been observed by Pachner [J. Acoust. Soc. Amer. 23, 185–198 (1951); these Rev. 12, 876] and earlier by Diemer [Thesis, Groningen, 1943, Stelling VII]. *C. J. Bouwkamp.*

Riekstyn'š, Ė. Ya. On some special functions applicable to the solution of the telegraph equations. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 485–494 (1951). (Russian)

The functions are $u_n(x,y)$, $v_n(x,y)$, where

$$u_n(x,y)=\sum_{m=0}^{\infty}(\sqrt{x/y})^{n+2m}I_{n+2m}(2\sqrt{xy}),$$

and I_ν denotes Bessel's function of imaginary argument; for $v_n(x,y)$, $\sqrt{(y/x)}$ replaces $\sqrt{(x/y)}$. The functions are a modification of Lommel's functions. The author gives first some twenty identities and differential and integral relations

satisfied by the functions. Next, on the basis of the formula $\partial^2 u_\nu/\partial x\partial y - u_\nu = 0$, he gives some solutions of the ordinary (non-matrix) telegraphic equation. Finally, on the basis of the Laplace transform, he gives asymptotic formulas for large x and y. In particular, for $\nu = 0, 1$, $\lim e^{-(\alpha+\beta)x} u_\nu(\alpha x, \beta x) = \frac{1}{2}, \frac{1}{4}$ or 0 according as $\alpha > \beta$, $\alpha = \beta$, or $\alpha < \beta$. The whole investigation parallels work of P. I. Kuznecov [Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 267–270 (1947); 11, 555–560 (1947); 12, 141–148 (1948); these Rev. 9, 30, 283; 10, 90], who considered a different modification of Lommel's functions. The theory of the present functions is claimed to be simpler.

F. V. Atkinson (Ibadan).

Lauwerier, H. A. Poiseuille functions. Appl. Sci. Research A. 3, 58–72 (1951).

The author defines two linearly independent solutions for each of the two differential equations

$$\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} \pm 4\omega^2(1-r^2)y = 0.$$

The four functions correspond, roughly, to the four Bessel functions J, Y, I, K. He expresses his functions in terms of Whittaker's confluent hypergeometric functions, gives power series expansions and integral representations, and uses the method of steepest descents to obtain asymptotic expansions of his functions for large ω, and $0 < r < 1$.

A. Erdélyi (Pasadena, Calif.).

The oscillating wing in a subsonic flow. Computation Dept., Math. Centrum, Amsterdam, Rep. R 53, Int. 1. i+7 pp. (5 tables) (1949); Rep. R 53, Int. 2. i+21 pp. (1949); Rep. R 53, Int. 3. i+2 pp. (17 tables) (1950); Rep. R 53, Int. 4. i+12 pp. (1950); Rep. R 53, Int. 6. i+2 pp. (41 plates) (1951); Rep. R 53, Int. 7. i+35+3 pp. (1951); Rep. R 53, Int. 8. i+2 pp. (17 tables) (1951).

Expansions of $B_n^{(n)}$ and a_n into power-series with respect to t. Computation Dept., Math. Centrum, Amsterdam, Rep. R 53, Int. 5. i+12 pp. (1950).

These interim reports give information about work carried out by the Computation Department of the (Amsterdam) Mathematical Centre on behalf of the National Aeronautical Research Institute in Amsterdam. They concern largely Mathieu functions, and also certain integrals related to Bessel functions. For Mathieu functions the notations of McLachlan's book [Theory and application of Mathieu functions, Oxford, 1947; these Rev. 9, 31] are adopted.

The theoretical part of Int. 1 establishes series expansions of the integrals

$$P_n(\beta^2\Omega, \eta_1, \eta_0) = \frac{1}{\pi}\int_0^{\eta_0} \exp(i\beta^2\Omega\cos\eta) \times \sin(n\eta)\log\frac{1-\cos(\eta+\eta_1)}{1-\cos(\eta-\eta_1)}d\eta, \quad n = 1, 2$$

$$R_n(\beta^2\Omega, \eta_1) = \int_0^{\eta_1} \exp(i\beta^2\Omega\cos\eta)\cos(n\eta)d\eta, \quad n = 0, 1, 2, 3, \cdots.$$

The cases $\eta_0 = \pi$ and $\eta_0 = \eta_1$ are of principal interest. Numerical tables of certain Bessel functions and trigonometric functions occurring in the expansions are added. They have been used in constructing tables of R_n for $n = 0(1)20$, $\frac{1}{2} - \frac{1}{2}\cos\eta_1 = .1(.05).35$, and varying ranges of $\beta^2\Omega$ between 0 and 4.8. The tables of R_n form Int. 6.

The odd periodic Mathieu functions are of the form

$$\mathrm{se}_n(x, q) = \sum B_m^{(n)} \sin mx$$

where m runs through even or odd integers according as n is even or odd. The corresponding characteristic value is b_n. The $B_m^{(n)}$ satisfy recurrence relations, and these are used in Int. 2 for the expansion of $B_m^{(n)}$ into an inverse factorial series in m. Expansions of b_n and $B_m^{(n)}$ in powers of q are given in Int. 5. Numerical tables of the characteristic values b_n for $k = q^{\frac{1}{2}}$ between .025 and 3 and $n = 1(1)12$, and of the coefficients $B_m^{(n)}$ for k between .025 and 3, $n = 1(1)8$, and varying ranges of m up to 33, are given in Int. 3. Further in Int. 2, the non-periodic solution of Mathieu's equation is written in the form

$$\mathrm{ge}_n(x) = x\,\mathrm{se}_n(x) + \sum g_m^{(n)} \cos mx,$$

and its coefficients $g_m^{(n)}$ are also expanded in an infinite series (which in this case involves the logarithmic derivative of the gamma function as well as inverse factorials.) Formulas are also developed for the computation of Mathieu functions of the third kind, $\mathrm{Ne}_n^{(3)}(x)$.

In Int. 4 and Int. 7, series expansions are obtained for the integral

$$\int_0^\infty e^{i\Omega\cosh t}\sinh(\nu t)dt$$

and for other integrals related to Bessel functions, and these expansions are used in the computation of certain integrals and infinite series involving Mathieu functions.

The last of these reports, Int. 8, contains numerical tables of the integrals

$$\int_0^x J_0(t)\begin{matrix}\cos\\ \sin\end{matrix} ctdt, \quad \int_0^x Y_0(t)\begin{matrix}\cos\\ \sin\end{matrix} ctdt, \quad \int_0^\infty \frac{e^{-ix\cosh t}}{\cosh t + c}dt$$

for $c = .3(.1).8$, $x = 0(.1)6.1$.

A. Erdélyi.

Bagchi, Hari Das, and Chakrabarti, Nalini Kanta. Note on certain integrals and series involving Tschebyscheff's functions. Bull. Calcutta Math. Soc. 43, 37–40 (1951).

"Tschebyscheff's functions" are $T_n(z) = 2^{1-n}\cos(n\cos^{-1}z)$ and $V_n(z) = 2^{-n}[z - (z^2-1)^{\frac{1}{2}}]^n$. The inequality

$$|V_n(z)| \leqq 2^{-n-\frac{1}{2}}(n\pi)^{\frac{1}{2}}(|z|^2-1)^{-n/2},$$

said to hold for $|z| > 1$ when n is very large, is a fair sample of the results presented in this paper. It is derived from an integral representation of $V_n(z)$.

A. Erdélyi.

Differential Equations

Germay, R. H. Sur les intégrales infiniment voisines des systèmes d'équations récurro-différentielles. Bull. Soc. Roy. Sci. Liège 20, 238–248 (1951).

Germay, R.-H. Sur les dérivées, par rapport aux paramètres, des intégrales infiniment voisines des systèmes d'équations différentielles récurrentes. Généralisation des équations aux variations de Poincaré. Bull. Soc. Roy. Sci. Liège 20, 333–346 (1951).

These papers prove that the determined system

$$y'_{ij} = f_{ij}(x, y_{j,l+1}, \lambda_m), \quad y_{ij}(0, \lambda_m) = \theta_{ij}(\lambda_m),$$

$(i, j = 1, \cdots, p; k = 1, \cdots, \infty; l = 0, \cdots, q; m = 1, \cdots, s)$ has a solution. In the first, continuity and Lipschitz conditions imposed on the data imply continuity of the solution and in the second continuous differentiability of the data implies the same property for the solution.

J. M. Thomas.

Amerio, Luigi. Sull'estensione delle nozioni di "colle," "nodo" e "fuoco" ai sistemi di due equazioni differenziali periodiche, in tre variabili. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 289–297 (1951).

This is a continuation of an earlier note [same journal (8) **10**, 206–212 (1951); these Rev. **12**, 827]. The analogies which were indicated in the earlier note are here discussed more fully, particularly in their geometrical aspects. The result is a rather satisfactory description of the qualitative properties of the solutions of a system of equations of the form

$$dx/dt=X(t,x,y),\quad dy/dt=Y(t,x,y),$$

where X and Y are periodic with respect to t with the common period T, in the neighborhood of a solution for which x and y are also periodic with the period T.

L. A. MacColl (New York, N. Y.).

Popov, Kiril. On the theory of linear differential equations. Doklady Akad. Nauk SSSR (N.S.) **79**, 17–20 (1951). (Russian)

The author illustrates, using the system $dx/dt=ax+by+c$, $dy/dt=kx+ly+m$ (a, b, c, k, l, m are functions of t), a formal method of integrating a linear system. The method consists of combining the variation of constants formula with an interchange of the order of integration to obtain a Volterra type integral equation, on which an iteration scheme can be applied. *E. A. Coddington* (Cambridge, Mass.).

Leavitt, William G. On systems of linear differential equations. Amer. J. Math. **73**, 690–696 (1951).

The paper is concerned with the linear system of differential equations

$$\mathbf{U}'=\{\lambda A+P\}\mathbf{U}$$

in which $\mathbf{U}$ is a vector and A and P are square matrices of order n. The elements of A are analytic functions of z in a region R. With a unimodular matrix T the transformation $\mathbf{U}=T\mathbf{U}_1$ changes the equation into a similar one in which the counterpart of A can be made triangular. It is shown that this is so even if R is unbounded. For the case $n=2$ it is shown that a canonical form is obtainable, in which the counterpart of A is of the form

$$\begin{pmatrix}\mu & \psi\\ 0 & -\mu\end{pmatrix}$$

with ψ either 0 or a proper divisor of μ. The main purpose of the paper is that of obtaining asymptotic solutions of the system relative to λ, when R is bounded and μ has just one zero in R. In this case ψ in the canonical form is either 1 or 0. The case $\psi=1$ permits transformation of the system until its asymptotic solutions are obtainable from known results (of the reviewer) for a scalar differential equation $y''+\{\lambda^2\mu^2(z)+\lambda\chi_1(z)+\chi_2(z,\lambda)\}y$. The case $\psi=0$ is dealt with directly. Formal solutions are derived, and under appropriate hypotheses their convergence is established.

R. E. Langer (Madison, Wis.).

Keller, Herbert B., and Keller, Joseph B. On systems of linear ordinary differential equations. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. **EM-33**, iii+28 pp. (1951).

The paper deals, for the most part, with formal expressions for the solutions of a system $U'=A(z)U$. Under rather heavy conditions on $A(z)$, an asymptotic formula for solutions of $U'=kA(z)U$, as $k\to\infty$, is found. *P. Hartman.*

Saharnikov, N. A. Solution of the problem of the center and the focus in one case. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 651–658 (1950). (Russian)

The author considers the system of equations

$$dx/dt=y+F(x,y),\quad dy/dt=-x-G(x,y),$$

where F and G are homogeneous polynomials of the third degree and gives the necessary and sufficient conditions that $x=0$, $y=0$ represent a focus or a center. *R. Bellman.*

Četaev, N. G. On the choice of parameters of a stable mechanical system. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 371–372 (1951). (Russian)

A system of linear differential equations

$$(*)\qquad dx_s/dt=\sum_r p_{sr}x_r\qquad (s,r=1,\cdots,n)$$

is considered, where the p_{sr} are constants depending on certain parameters. Under certain assumptions on these parameters, an upper bound for the time of travel of a point $(x_1,\cdots,x_n)$ on a trajectory of (*) from the sphere $x_1^2+\cdots+x_n^2=A$ to the sphere $x_1^2+\cdots+x_n^2=\epsilon$ is obtained. A condition on the parameters is given for which this upper bound is a minimum. *E. A. Coddington.*

Malkin, I. G. On a method of solution of the problem of stability in the critical case of a pair of purely imaginary roots. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 473–484 (1951). (Russian)

A real system

$$x'=\lambda y'+X(x',y'),\quad y'=-\lambda x'+Y(x',y'),$$

where X, Y are analytical at the origin and begin with second degree terms at least may be put in the form

$$\dot{x}=i\lambda x+Z(x,y),\quad \dot{y}=-i\lambda x+Z(y,x)$$
$$x=x'+iy',\quad y=x'-iy'.$$

The author applies a transformation

$$x=u+x^{(2)}(u,v)+x^{(3)}(u,v)+\cdots,$$
$$y=v+y^{(2)}(u,v)+y^{(3)}(u,v)+\cdots,$$

where $x^{(j)}$, $y^{(j)}$ are forms of degree j to reduce the system to

$$\dot{u}=i\lambda u+u(A_1uv+A_2u^2v^2+\cdots+A_{2n+1}u^{2n}v^{2n}+\cdots),$$
$$\dot{v}=-i\lambda+v(A_1uv+\cdots),$$

where the A_j are constants, and thus succeeds in simplifying greatly the Liapunov system for stability. Let A_k be the first constant which is not pure imaginary: $A_k=g+iB_k$, $g\neq 0$. It is the same g as occurs in the calculations of Lyapunov. If $g<0$ the system is stable, if $g>0$ it is unstable. An analogous treatment is applied to a system of n equations with a pair of pure imaginary characteristic roots and shown to simplify computations considerably. [Relevant references: Lyapounoff, Problème général de la stabilité du mouvement, Ann. of Math. Studies, no. 17, Princeton Univ. Press, 1947; Malkin, Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 255–257 (1951); these Rev. **9**, 34, **13**, 38.]

S. Lefschetz (Princeton, N. J.).

Malkin, I. G. Solution of some critical cases of the problem of stability of motion. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 575–590 (1951). (Russian)

In the paper reviewed above the author gave a simplified method for settling the problem of stability of a system

$$\frac{dx}{dt}=-\lambda y+X,\quad \frac{dy}{dt}=\lambda x+Y,$$

$$\frac{dx_s}{dt}=\sum p_{sj}x_j+p_{sx}x+q_sy+X_s,\quad s=1,2,\cdots,n,$$

where X, Y, X_s are analytic in x, y, and the x_i in the neighborhood of the origin and begin with terms of degree $\geqq 2$ and where the characteristic roots of $\|p_{ij}\|$ have negative real parts. The same general type of simplification is now extended to a similar system

$$\frac{dy_j}{dt}=\sum_{h=1}^{k} q_{jh}y_h+Y_j, \quad j=1, 2, \cdots, k,$$
$$\frac{dx_s}{dt}=\sum p_{sr}x_r+X_s, \quad s=1, 2, \cdots, n,$$

under the following two hypotheses: (a) $k=4$ and the characteristic roots of q_{jh} are purely complex; (b) $k=3$ and the characteristic roots of $\|q_{jh}\|$ are zero and $\pm\lambda i$. It is still assumed that the characteristic roots of $\|p_{sr}\|$ have negative real parts. The method is also extended to the case where the coefficients of the Y_j and X_s are periodic in t with the same period. [Additional references: G. V. Kamenkov, Sbornik Trud. Kazan. Aviac. Inst., no. 9 (1939); Malkin, Akad. Nauk SSSR. Prikl. Mat. Meh. 6, 411–448 (1942); these Rev. 4, 225; Mat. Sbornik N.S. 3(45), 47–101 (1938).]

S. Lefschetz (Princeton, N. J.).

Luzin, N. N., and Kuznecov, P. I. On absolute invariance and ϵ-invariance in the theory of differential equations. III. Doklady Akad. Nauk SSSR (N.S.) **80**, 325–327 (1951). (Russian)

The system of differential equations (1) $\sum_{j=1}^{n} a_{ij}x_j=\delta_i^1 f(t)$, where a_{ij} is a polynomial with constant coefficients and of second degree in D, $D\equiv d/dt$, is considered under the assumption that $\Delta(D)=|a_{ij}|$ is not identically zero. Let H be the maximum of the coefficients of the polynomial in D defined by the minor of a_{11} in $\Delta(D)$. Assume that $f(t)$ and the first $2n-2$ derivatives of $f(t)$ are in absolute value less than M on $-\infty<t<\infty$. In the present paper the authors emphasize some of their earlier results [C. R. (Doklady) Acad. Sci. URSS (N.S.) **51**, 251–253, 335–337 (1946); these Rev. **8**, 73]. If $(E_1(t), \cdots, E_n(t))$ is a bounded solution of (1), then $|E_1(t)|\leqq CHM$, where C depends only on the coefficients of the determinant $\Delta(D)$. In the present paper it is pointed out that given $\epsilon>0$, one can adjust $\Delta(D)$ so that $CHM<\epsilon$, hence $|E_1(t)|$ is uniformly small. If $f(t)$ is periodic but not identically zero, and if N is any given number, there exists a $\Delta(D)$ such that $E_1(t)>N$ for some t.

F. G. Dressel (Durham, N. C.).

Prodi, Giovanni. Nuovi criteri di stabilità per l'equazione $y''+A(x)y=0$. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 447–451 (1951).

It was shown by Ascoli that all solutions of (1) $u''+(1+a(t))u=0$ are bounded if $a(t)$ is the sum of a function of bounded variation over $[t_0, \infty]$ and a function which is absolutely integrable over $[t_0, \infty]$. Subsequently it was shown by the same author that if $a(t)=t^{-1}\sin \alpha t$, a function which escapes the previous criteria, then the solutions are bounded if $\alpha\neq 2$, and unbounded otherwise. In the present note it is shown that the solutions of $u''+(1+t^{-\sigma}\sin \alpha t)u=0$ are bounded if $\alpha\neq 2$ and $\sigma>\frac{1}{2}$. This is a special case of a general result obtained by converting (1) into an integral equation $u=v+\int_0^t k(t, t_1)a(t_1)u(t_1)dt_1$, where v is the general solution of $u''+u=0$, and iterating once, obtaining

$$u=v+\int_0^t k(t, t_1)a(t_1)v(t_1)dt_1 + \int_0^t k(t, t_1)a(t_1)\left(\int_0^{t_1} k(t_1, t_2)a(t_2)k(t_2)dt_2\right)dt_1.$$

R. Bellman (Stanford University, Calif.).

van Kampen, N. G. Note on the analytic continuation of the S-matrix. Philos. Mag. (7) **42**, 851–855 (1951).

The author discusses the equation

$$u''(r)+\{E-V(r)\}u(r)=0,$$

with the boundary condition $u(0)=0$, and the problem of determining the discrete spectrum for this situation from the Heisenberg S-matrix. It is known [S. T. Ma, Physical Rev. (2) **71**, 195–200 (1947); these Rev. **8**, 425] that the determination of the eigenvalues from the S-matrix may be impossible for every potential $V(r)$, because of the possibility of certain "redundant poles" in the S-matrix. The author's summary says: "It is shown that methods which try to avoid the redundant poles are not likely to be successful. On the other hand, the analytic continuation of the S-matrix is an ill-defined procedure, which cannot be applied until more properties of the S-matrix are known. The redundant poles are caused by the use of the asymptotic expression of the wave function instead of its actual value and have no physical meaning." Nowhere in the paper is mention made of what type of potential $V(r)$ is being considered, and the arguments proposed definitely depend on the nature of $V(r)$. The reviewer would like to point out that Kodaira [Amer. J. Math. **71**, 921–945 (1949), especially pp. 937–944; these Rev. **11**, 438] has previously considered these questions from a rigorous mathematical point of view.

E. A. Coddington (Cambridge, Mass.).

Voronov, A. A. Free oscillations of an oscillator with variable friction. Doklady Akad. Nauk SSSR (N.S.) **81**, 517–520 (1951). (Russian)

The equation under discussion is of the general type

$$(1)\qquad \begin{cases} \ddot{x}+\omega_0^2 x=-k^2 f(x)\ \text{sign}\ \dot{x}, \\ f=f_0+|f_1|, \\ f_0=\sum_0^n a_{2k}x^{2k}, \quad f_1=\sum_0^n a_{2k+1}x^{2k+1}. \end{cases}$$

It represents a complicated motion under friction (dry friction when f is constant). In the four quadrants f assumes the values of four distinct polynomials; In each quadrant the phase trajectories belong to one of the two families

$$(2)\qquad \dot{y}^2+\omega_0^2 x^2\pm 2k^2\int f(x)dx=C.$$

Special attention is paid to the case (3) $f_0=0$, $f_1(x)=x$ when the families (2) are ellipses (since $k\ll\omega_0$). The full trajectories spiral then around the origin. There is a "logarithmic decrement" which for small k is $S\cong 1+2k^2/\omega_0^2$ and the period of the oscillations is then $T\cong 2\pi/\omega_0$ (as might be expected). It is of interest with variable frictions to replace (3) by the linearized system

$$\ddot{x}+2h_k\dot{x}+\omega_k^2 x=0,$$

with equal approximate period and logarithmic decrement as (3). The same method is applicable to a system with viscous friction

$$\ddot{x}+2h\dot{x}+\omega_0^2 x=-k^2|x|\,\text{sign}\,\dot{x}.$$

[Reference: Andronow and Chaikin, Theory of oscillations, Princeton Univ. Press, 1949, pp. 107–131; these Rev. **10**, 539].

S. Lefschetz (Princeton, N. J.).

Železcov, N. A. The method of point transformation and the problem of the forced vibrations of an oscillator with "combined friction." Amer. Math. Soc. Translation no. 57, 49 pp. (1951).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 3–40 (1949); these Rev. 10, 607.

de Castro Brzezicki, A. Study and solution of the differential equation $xy''+ny'+axy=0$. Gaceta Mat. (1) 3, 153–155 (1951). (Spanish)

It is shown that the solutions of the equation of the title can be expressed explicitly in terms of elementary functions when n is an even integer, and in terms of elementary functions and Bessel functions when n is an odd integer.

L. A. MacColl (New York, N. Y.).

Krein, M. G. On certain problems on the maximum and minimum of characteristic values and on the Lyapunov zones of stability. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 323–348 (1951). (Russian)

In the first part of the paper are determined precise upper and lower bounds for the squares of the vibrational frequencies of a string of prescribed length l and total mass M, raised to unit tension, whose density-function $\rho(x)$ $(0\leqq x\leqq l)$ is subject to $0\leqq\rho(x)\leqq H$, for given $H>M/l$. The corresponding density-functions are also found; the lowest frequency for example is minimised by concentrating the mass M towards the middle of the string, and maximised by concentrating it towards the ends. Results for higher frequencies are deduced from those for the first by the device of supposing the string pinned down at the nodes. Making $H\to\infty$ the author's results include one due to I. M. Rapoport [Doklady Akad. Nauk SSSR (N.S.) 73, 889–890 (1950); these Rev. 12, 183]. Next various generalisations are summarised. Full results are stated, without proof, for the case in which $\rho(x)$ has a positive lower bound as well as an upper bound. The more general differential equation $y''-q(x)y+\lambda\rho(x)y=0$ is noted, and appears to involve no new feature. Corresponding problems for vibrating membranes are stated to be more difficult; however the case of a circular membrane is solved in part. In the next section the foregoing results are applied to deduce approximate formulae for the first characteristic value of a non-uniform string in terms of its moment of inertia about the mid-point. Finally the author passes to Liapounoff zones of stability, i.e. ranges of values of λ such that the solutions of $y''+\lambda\rho(x)y=0$ are bounded on the whole real axis, where $\rho(x)$ is periodic of period l. One of his results is that the first stability zone includes the range

$$0<\lambda<(4/l)\left(\int_0^l \rho_+(x)dx\right)^{-1},$$

where $\rho_+(x)=\rho(x)$ $(\rho(x)\geqq 0)$, $=0$ $(\rho(x)\leqq 0)$. This improves a result of G. Borg [Amer. J. Math. 71, 67–70 (1949); these Rev. 10, 456]. There are several other results for this and subsequent stability zones. *F. V. Atkinson.*

Kodaira, Kunihiko. On singular solutions of second order differential operators. I. General theory. Sūgaku (Mathematics) 1, 177–191 (1948). (Japanese)

Kodaira, Kunihiko. On singular solutions of second order differential equations. II. Applications to special problems. Sūgaku (Mathematics) 2, 113–139 (1949). (Japanese)

A more detailed exposition is given of the author's paper in Amer. J. Math. 71, 921–945 (1949); these Rev. 11, 438.

K. Yosida (Nagoya).

Miller, Kenneth S. Self-adjoint factorizations of differential operators. Proc. Amer. Math. Soc. 2, 704–705 (1951).

The author proves that if, in the formal differential operator $L=\sum_{k=0}^{2r}p_k(x)d^{2r-k}/dx^{2r-k}$, the coefficient p_k is of class C^{2r-k} and $p_0(x)>0$ for $a\leqq x\leqq b$, then, on some subinterval, L has the factorization $L=f(x)D_1\cdots D_r$, where D_r is a formal self-adjoint differential operator of second order.

P. Hartman (Baltimore, Md.).

Miller, Kenneth S. The one-sided Green's function. J. Appl. Phys. 22, 1054–1057 (1951).

Let L be an nth order linear differential operator defined in a half-interval $[a, \infty)$. Then under certain conditions there is a unique integral kernel $H(x, \xi)$ such that if $u(x)=\int_a^x H(x, \xi)f(\xi)d\xi$, then $Lu(x)=f(x)$ and $u^{(\alpha)}(a)=0$, for $\alpha=0, 1, \cdots, n-1$. The most novel of the author's results seems to be a process for deriving from $H(x, \xi)$ a fundamental system of solutions of $Lu=0$. It is observed that $H(x, \xi)$ can be visualised as the impulsive response of a linear network. [Reviewer's remark: The concept of such a function appears to owe more to Cauchy than to Green; see E. Goursat, Cours d'analyse mathématique [2nd ed. Gauthier-Villars, Paris, 1911], tome II, pp. 421–422].

F. V. Atkinson (Ibadan).

Adem, Julián, and Moshinsky, Marcos. Self-adjointness of a certain type of vectorial boundary value problems. Bol. Soc. Mat. Mexicana 7, 1–17 (1950). (Spanish)

Let $y(x)$ denote a column matrix $(1\times n)$, $P(x)$, $Q(x)$, $R(x)$ and $W(x)$ square matrices $(n\times n)$, and A, B, C and D rectangular matrices $(n\times 2n)$. The paper contains a study of the system consisting of the differential equations

$$M(y)=Py''+Qy'+Ry=\lambda Wy,$$

with boundary conditions

$$Ay(0)+By(1)+Cy'(0)+Dy'(1)=0.$$

Definitions of exactness, the adjoint equation and system, etc., as well as much of the development, have close parallels in the scalar case [as in E. L. Ince, Ordinary Differential Equations, Longmans Green, London, 1927]. To illustrate the exposition, a system with $n=4$ is tested for self-adjointness.

R. E. Gaskell (Seattle, Wash.).

Adem, Julián, and Moshinsky, Marcos. On matrix boundary value problems. Quart. Appl. Math. 9, 424–431 (1952).

The temperature in a cross can be described by a set of functions satisfying a system of partial differential equations with appropriate boundary and initial conditions. With no loss of heat from the lateral surfaces, and if the variation of temperature throughout the intersection of the cross bars is neglected, flow of heat becomes one-dimensional in any of the four bars. The intersection being small, its heat capacity is also neglected, and a separation of variables is then possible. In the course of the solution, the authors apply their theory developed elsewhere [see the paper reviewed above] to solve the characteristic value problem arising. A second illustration concerns the vibration of two circular plates with an intermediate elastic medium.

R. E. Gaskell (Seattle, Wash.).

✓*Leimanis, Eugene. **The application of infinitesimal transformations to the integration of differential equations of exterior ballistics by quadratures.** Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 206–217. University of Toronto Press, Toronto, 1951. $6.00.

The projectile is considered a material point acted on by only two forces, gravity and drag, the latter being in the tangent to the trajectory. The function in the hodograph equation is $f(v)$ and the density of the air is $\delta(y)$. The Lie theory is applied to show that in the following cases the variables can be separated: $f=v$, $\delta=(1+by)^n$ for

$$n=-2,\ -\tfrac{1}{2},\ 0,\ 1;\quad f=v,\ \delta=e^{kv},\ f=v^2,\ \delta=(1+ky)^{-1};$$
$$f^{-1}=v^2\exp(av^2/2g);\quad \delta=\exp(-ay).$$

It is said that these include all known and some new integrable cases. *J. M. Thomas* (Durham, N. C.).

Herbst, Robert Taylor. Passive total systems with constant coefficients. Proc. Nat. Acad. Sci. U. S. A. **37**, 710 (1951).

The system

$$dY=A_iYdx^i,\quad Y(0)=Y_0,\quad A_iA_j=A_jA_i,\quad (i,j=1,2,\cdots,r),$$

where A_i are known constant square matrices, Y is an unknown square matrix, Y_0 is a known square matrix and i is summed, has the unique solution $Y=Y_0\exp(A_ix^i)$, which can be found by König's integration process. Cf. also Saltykow, C. R. Acad. Sci. Paris **225**, 520–521 (1947); Vessiot, ibid. **226**, 289–291 (1948); Pini, Boll. Un. Mat. Ital. (3) **5**, 255–264 (1950); these Rev. **9**, 186, 354; **12**, 708.

J. M. Thomas (Durham, N. C.).

Erdélyi, A. The analytic theory of systems of partial differential equations. Bull. Amer. Math. Soc. **57**, 339–353 (1951).

La théorie analytique des équations différentielles linéaires a été, comme on sait, étendue dans une certaine mesure à certaines équations aux différentielles totales linéaires par les travaux d'Appell et de Picard (1880), de Goursat (1882), de Horn (1889–1893). Il est naturel d'espérer que cette extension pourrait être reprise à l'heure actuelle avec quelques chances de succès grâce aux progrès réalisés dans la théorie des fonctions de plusieurs variables complexes ainsi qu'en géométrie algébrique et en topologie. Le présent article vise à préparer cette extension: il rappelle les points essentiels de la théorie des équations différentielles linéaires, en particulier du type de Fuchs, et les théorèmes d'existence des solutions des systèmes d'équations aux dérivées partielles; puis il précise, dans le cas des systèmes aux différentielles totales linéaires à deux variables independantes, à coefficients rationnels, ce qu'on doit entendre par courbe singulière du type régulier. En se bornant aux systèmes d'Appell, satisfaits par certaines séries hypergéométriques à deux variables, l'auteur indique les résultats précis auxquels il est parvenu [Acta Math. **83**, 131–164 (1950); ces Rev. **12**, 257]: le voisinage d'un point singulier doit être partagé, par exemple, en trois parties, et c'est pour les courbes fermées intérieures a chacune de ces trois parties que l'on peut préciser la forme d'un système canonique fondamental, cette forme changeant d'une partie à une autre. D'autre part un système considéré par Horn lui donne occasion de réduire, grâce à une transformation birationnelle, le type d'une singularité; il semble possible de réduire ainsi, en général, chaque singularité d'ordre élevé en singularités élémentaires; mais l'auteur fait remarquer que rien ne garantit que cela puisse etre réalisé simultanément pour tous les points singuliers.

On sait que Riemann avait pu déduire les propriétés de la fonction hypergéométrique uniquement de certaines hypothèses générales (nature au voisinage des points singuliers . . .) et que Picard et Goursat ont donné d'intéressantes extensions de cette idée aux fonctions hypergéométriques de deux variables. Les idées de Riemann ont conduit à la théorie de Fuchs sur les équations différentielles à singularités régulières; une théorie correspondante pour les systèmes d'équations aux dérivées partielles ne semble possible à l'heure actuelle en raison du manque de critères convenables pour les "courbes singulières de type régulier."

M. Janet (Paris).

*John, Fritz. **General properties of solutions of linear elliptic partial differential equations.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 113–175. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

The paper is an abstract of a series of lectures. Put $D_i=\partial/\partial x_i$ $(1\leqq i\leqq n)$ and let $L=L(D_1,\cdots,D_n)$ be a linear differential operator of order m whose coefficients are real or complex analytic functions of the real variable $x=(x_1,\cdots,x_n)$. The homogeneous polynomial of degree m,

$$Q(\xi)=(m!)^{-1}d^mL(t\xi_1,\cdots,t\xi_n)/dt^m$$

of the real variable $\xi=(\xi_1,\cdots,\xi_n)$ is called the characteristic form of L and depends on x. In chapter 1 [see John, Comm. Pure Appl. Math. **2**, 209–253 (1949); these Rev. **12**, 185] the author extends the method of Holmgren [Öfversigt af Kongl. Vetenskaps-Akademiens Förhandl. **58**, 91–103 (1901)] to prove certain uniqueness theorems concerning Cauchy's problem for the equation $Lu=0$. It is shown that the Cauchy data of u on a non-characteristic analytic hypersurface S determine u uniquely in regions attainable by analytic deformation with fixed boundary of S preserving its property of being non-characteristic, i.e. such that $Q(\xi)\neq 0$ on S, ξ being the normal of S. In chapter 2 [see John, Comm. Pure Appl. Math. **3**, 273–304 (1950); these Rev. **13**, 40] a fundamental solution of the equation $Lu=0$ is constructed under the assumption that L is elliptic so that $Q(\xi)\neq 0$ for every x and all $\xi\neq 0$. The ingenious construction which is connected with Holmgren's method of proving uniqueness, is surprisingly simple and a major contribution to the theory of elliptic partial differential equations. In chapter 3 the construction is extended by the method of E. E. Levi to elliptic operators with sufficiently differentiable coefficients. The fourth chapter contains some results concerning the differentiability of solutions of non-analytic elliptic partial differential equations derived by the method of spherical means. *L. Gårding* (Lund).

Heins, Albert E. Some remarks on the coupling of two ducts. J. Math. Physics **30**, 164–169 (1951).

The mathematical problem of the coupling of two ducts is dependent on the solution of $\phi_{xx}+\phi_{yy}-k^2\phi=0$ subject to the conditions $\phi_y=0$ for $y=0$, $-\infty<x<\infty$, $\phi_y=\sigma_1\phi$ for $y=a$, $x<0$, $\phi_y=\sigma_2\phi$ for $y=a$, $x>0$. The constants k, σ_1, σ_2 may be real or complex. Problems of this type have been considered by Feshbach and Heins [J. Math. Physics **26**, 143–155 (1947); these Rev. **9**, 255] and by Heins [Amer. J. Math. **70**, 730–748 (1948); these Rev. **10**, 490], in both cases by solving an integral equation of Wiener-Hopf type.

In a recent paper Weitz and Keller [Comm. Pure Appl. Math. **3**, 305–318 (1950); these Rev. **12**, 762], in discussing

the effect of surface waves on two coupled channels of finite depth, claimed that the Wiener-Hopf technique is inapplicable. It is shown here that, if the appropriate Green's function is used, the problem can be made to depend on a Wiener-Hopf integral equation, and the solution is carried out. *E. T. Copson* (St. Andrews).

Malyužinec, G. D. On focussing in an absorbing medium. Doklady Akad. Nauk SSSR (N.S.) **78**, 229–230 (1951). (Russian)

A "focus" of a solution w of $\Delta w+\lambda w=0$ is a point at which $|w|$ has a maximum. It is proved that no focus can exist if $\Re(\lambda)<0$. An example is given (from which the reviewer dissents) to show that a focus may exist if $\Re(\lambda)=0$, $|\lambda|>0$. Also considered is whether $|w|$ can have a maximum on a boundary, the boundary condition being $\partial w/\partial n+hw=0$, where n is the outward normal. This is shown to be impossible if everywhere $\Re(h)>0$, amongst other conditions. *F. V. Atkinson* (Ibadan).

Martin, M. H. The fundamental solution of $\Delta\psi+e(y)\psi_y=0$. Duke Math. J. **18**, 845–858 (1951).

The fundamental solution of

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+e(y)\frac{\partial\psi}{\partial y}=0 \tag{1}$$

with singularity (x_0, y_0) is a solution of the form

$$\psi=P(x, y)\log r+W(x, y),$$

where $P(x_0, y_0)\neq 0$, $r^2=(x-x_0)^2+(y-y_0)^2$, in which P and W are regular at the singularity. An explicit formula for the fundamental solution in the case $e=p/y$ was given by Tricomi [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (5) **14**, 133–247 (1923)] for $p=1/3$ and by Weinstein [Trans. Amer. Math. Soc. **63**, 342–354 (1948); these Rev. **9**, 584]. The case of general $e(y)$ is discussed in the paper under review.

The method consists in introducing new variables $\xi=\xi(r, y)$, $\eta=y$, which transforms (1) into

$$\alpha\frac{\partial^2\psi}{\partial\xi^2}+2\beta\frac{\partial^2\psi}{\partial\xi\partial\eta}+\frac{\partial^2\psi}{\partial\eta^2}+\delta\frac{\partial\psi}{\partial\xi}+e\frac{\partial\psi}{\partial\eta}=0, \tag{2}$$

where the coefficients are, in theory, functions of ξ and η. Of course, $\xi(r, y)$ is an unknown function so far; but even if it were known, it would be very difficult to express the coefficients in (2) in terms of ξ, η. This difficulty is turned by a change of viewpoint. Given arbitrary functions α, β, δ of ξ, η, conditions are found for the existence of a function $\xi(r, y)$ such that (2) is the transformation of (1). It is assumed that $\alpha=X_2Y_2$, $\beta=Y_0+X_1Y_1$, $\delta=Y_3+X_1Y_4$, where the X_i are functions of ξ alone, the Y_i of η alone; and the resulting equation (2) is solved by a generalisation of the method of separation of variables. Conditions are found to ensure that (2) has a solution of the form $\psi=X(\xi)Y(\eta)$. This process, of which the details are rather complicated, leads to equations for the determination of e, X_i, Y_i, X, Y and ξ. There is a certain arbitrariness in the solution. It is shown that with one particular choice of the arbitrary constants which occur the method leads to the solution for

$$e(y)=\frac{p}{y}+\frac{2(1-p)}{y+Cy^p}\quad(p\neq 1)\qquad\text{or}\qquad e(y)=\frac{1}{y}+\frac{2}{y\log y+Cy}.$$

The results in the cases $p=0$ and $p=-2$ are particularly simple. *E. T. Copson* (St. Andrews).

Ciorănescu, Nicolae. Regular solution of a linear differential equation of the second order with variable coefficients. Acad. Repub. Pop. Române. Bul. Ști. A. **1**, 725–731 (1949). (Romanian. Russian and French summaries)

Ciorănescu, Nicolae. On representations by a simple series of a regular solution of Green's equation. Acad. Repub. Pop. Române. Bul. Ști. Ser. Mat. Fiz. Chim. **2**, 293–297 (1950). (Romanian. Russian and French summaries)

In the first paper it is shown that a solution of the partial differential equation $u_{xx}+u_{yy}+(2p+1)y^{-1}u_y=0$, which is regular at $y=0$ and satisfies $u_y(x, 0)=0$, is represented by

$$u(x, y)=\int_0^1 U(x, ty)(1-t^2)^{p-\frac{1}{2}}dt$$

where U is a harmonic function and $U_y(x, 0)=0$, and in the second paper this solution is expanded into a series of Bessel functions. Some related partial differential equations are also mentioned. *A. Erdélyi* (Pasadena, Calif.).

Il'in, V. A. The representation of the source-function for a rectangle as a bilinear series of characteristic functions. Doklady Akad. Nauk SSSR (N.S.) **74**, 413–416 (1950). (Russian)

The author investigates the convergence of bilinear series associated with the wave equation for rectangular regions, such as $\sum u_{mn}^{(p)}$, where

$$u_{mn}^{(p)}=\frac{4}{ab}\sin\frac{\pi mx}{a}\sin\frac{\pi ny}{b}\sin\frac{\pi m\xi}{a}\sin\frac{\pi n\eta}{b}\left[\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\right]^{-p}.$$

Courant and Hilbert [Methoden der mathematischen Physik, Band I, p. 335, Springer, Berlin, 1924] stated, apparently erroneously, that for $p=1$ this series is absolutely convergent, (x, y) being excluded from a neighbourhood of (ξ, η). The present author gives a disproof of this, and also states an extension of the result to n dimensions. He further states without detailed proof a number of results for the conditional convergence of such series, from which is said to follow the validity of the bilinear formula for the Green's function for the rectangle. *F. V. Atkinson.*

Il'in, V. A. On the convergence of bilinear series of characteristic functions. Doklady Akad. Nauk SSSR (N.S.) **74**, 653–656 (1950). (Russian)

Let u_i, λ_i, $i=1, 2, \cdots$, be eigenfunctions, presumably normalised, and eigenvalues of the wave-equation in an arbitrary n-dimensional region G with homogeneous boundary conditions. The author proves results on the convergence of the series $\sum_{i=1}^{\infty}u_i(P)u_i(Q)\lambda_i^{-p}$ for various values of p, extending some of his previous results for the case in which G was a rectangle [see the preceding review]. The first theorem asserts absolute convergence with $p=\frac{1}{2}n+\epsilon$, uniformly if P and Q lie in some subregion G' of G. Convergence in mean, integrated with respect to Q, then follows with $p=\frac{1}{2}n+\epsilon$. Finally, if G is a "cylinder" with arbitrary cross-section and axis parallel to the x_n-axis, conditional convergence is said to hold with $p=\frac{1}{2}(n-1)+\epsilon$. The author does not mention the work of Carleman and Pleijel [see Å. Pleijel, Amer. J. Math. **70**, 892–907 (1948); these Rev. **10**, 301] which goes further in the cases $n=2, 3$. [The reviewer is not satisfied concerning the proof of the first theorem, only part of which is given; the author does not appear to have made any use of the boundary conditions.] *F. V. Atkinson* (Ibadan).

Prodi, Giovanni. Questioni di stabilità per equazioni non lineari alle derivate parziali di tipo parabolico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 365–370 (1951).

The problem of demonstrating the stability in the sense of Poincaré-Liapounoff of the solution $u=0$ of the non-linear parabolic equation $u_t=u_{xx}+u_{yy}+u_{zz}+f(x, y, z, t, u)$ where $u=a(x, y, z)$ at $t=0$ for $(x, y, z)\in R$, $u=0$ on B, the boundary of R, and $|f|/|u|\to 0$ as $u\to 0$, was raised by the reviewer [Trans. Amer. Math. Soc. **64**, 21–44 (1948); these Rev. **10**, 43] and solved for some simple regions, in particular for a rectangular parallelepiped. The author, using some very simple methods tracing back to Haar and Westphal, proves the stability for general bounded regions and for more general forms of the parabolic equation. He further shows that the condition $|f|/|u|\to 0$ may be replaced by the weaker one $|f(u_1)-f(u_2)|\leqq c_1|u_1-u_2|$, where c_1 is a constant which depends upon R. Although the question of the existence of the solution of the above equation is not discussed, the methods used in the paper are equally applicable to the treatment of this problem. *R. Bellman.*

Akulov, N. S. On a generalization of a problem of Kolmogorov, Petrovskiĭ, and Piskunov. Doklady Akad. Nauk SSSR (N.S.) **80**, 573–576 (1951). (Russian)

The problem mentioned in the title is the determination of the velocity of propagation of diffusion waves when the diffusion process is accompanied by an autocatalytic growth of the number of diffusing particles. This problem is generalized to the case when the diffusion process involves particles of different types and when these particles, by reacting with the surrounding medium, undergo cyclic transformations from any one of the given types to any other. The mathematical model of the problem consists of the solution of a system of linear partial differential equations with certain boundary conditions. The special contribution of this paper is the solution of the problem for the case when the diffusion and chain reaction process takes place within a cylindrical tube, and the waves are propagated along the tube. The velocity of propagation is shown to be a function of the radius. The theory developed here permits the determination of the more important characteristic properties of the transmission of chain reaction diffusion waves.

H. P. Thielman (Ames, Iowa).

Barenblatt, G. I., and Levitan, B. M. On a generalization of Poisson's formula from the theory of heat conduction. Doklady Akad. Nauk SSSR (N.S.) **79**, 917–920 (1951). (Russian)

The authors give a generalization of the integral formula of Poisson which permits them to construct solutions for the boundary value problems for the equation of heat conduction. The fact that the solution is representable in the form of a generalized integral imposes on it certain conditions not usually associated with the heat equation (in particular, the vanishing at infinity). As an example, they construct a solution of the equation $q(x)\partial T/\partial t=\partial^2 T/\partial x^2+Q(x, t)$ for the boundary conditions $T(0, t)\cos\alpha-x^{-1}T(0, t)\sin\alpha=\phi(t)$, $T(x, 0)=f(x)$. *C. G. Maple* (Washington, D. C.).

Brazma, N. A., and Myškis, A. D. The law of conservation of energy in the theory of generalized systems of telegraph equations. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 495–500 (1951). (Russian)

The authors consider the vector-matrix telegraphic system

(*) $$-\partial\mathbf{u}/\partial x=\mathbf{R}\mathbf{i}+\mathbf{L}\partial\mathbf{i}/\partial t,\quad -\partial\mathbf{i}/\partial x=\mathbf{G}\mathbf{u}+\mathbf{C}\partial\mathbf{u}/\partial t,$$

$\mathbf{R}$, $\mathbf{L}$, $\mathbf{G}$ and $\mathbf{C}$ being constant symmetric matrices, in a region of the form $0\leqq x\leqq l$, $0\leqq t<T$. In the first section (due to A. D. Myškis) there is set up the energy integral

$$\frac{d}{dt}\frac{1}{2}\int_0^l(\mathbf{C}\mathbf{u}\cdot\mathbf{u}+\mathbf{L}\mathbf{i}\cdot\mathbf{i})dx=-\int_0^l(\mathbf{G}\mathbf{u}\cdot\mathbf{u}+\mathbf{R}\mathbf{i}\cdot\mathbf{i})dx,\quad (0\leqq t<T),$$

subject to the boundary conditions $\mathbf{u}=0$ for $x=0$, $x=l$. The first deduction states in effect that if $\mathbf{C}$, $\mathbf{L}$, $\mathbf{G}$ and $\mathbf{R}$ are non-negative definite, and at least one of $\mathbf{C}$ and $\mathbf{G}$ positive definite, and likewise for $\mathbf{L}$ and $\mathbf{R}$, then the solution of (*) is unique if $\mathbf{u}$ and $\mathbf{i}$ are given for $t=0$, and $\mathbf{u}$ is given for $x=0$ and $x=l$. The second deduction states that the elementary solutions of (*), obtained by separating the variables, will be attenuated as $t\to\infty$ if $\mathbf{C}$, $\mathbf{L}$, $\mathbf{G}$ and $\mathbf{R}$ are non-negative definite, and at least one of $\mathbf{G}$, $\mathbf{R}$ is positive definite; this clarifies considerably an earlier investigation [N. A. Brazma, Doklady Akad. Nauk SSSR (N.S.) **69**, 313–316 (1949); these Rev. **11**, 297]. In the second section N. A. Brazma explains the physical significance of these criteria.

F. V. Atkinson (Ibadan).

Brazma, N. A. The complete hyperbolicity of a generalized system of telegraph equations. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 501–503 (1951). (Russian)

In the vector-matrix telegraphic system

$$-\partial\mathbf{u}/\partial x=\mathbf{R}\mathbf{i}+\mathbf{L}\partial\mathbf{i}/\partial t,\quad -\partial\mathbf{i}/\partial x=\mathbf{G}\mathbf{u}+\mathbf{C}\partial\mathbf{u}/\partial t,$$

let $\mathbf{R}$, $\mathbf{L}$, $\mathbf{G}$, $\mathbf{C}$ be constant, symmetric, non-negative definite matrices. The condition for the complete hyperbolicity of this system is reduced to the requirement that the characteristic numbers of the matrix $\mathbf{LC}$ should be all positive. The latter is shown to be the case if $\mathbf{L}$ and $\mathbf{C}$ are both positive definite. One proof of this depends on the work of the paper reviewed above, another, attributed to M. A. Naĭmark, is purely algebraical. *F. V. Atkinson* (Ibadan).

Garnir, H. G. Sur le problème de Cauchy dans la théorie des distributions. Bull. Soc. Roy. Sci. Liège **20**, 86–95 (1951).

Let $L(\partial/\partial x_1, \cdots, \partial/\partial x_n, \partial/\partial t)$ be a differential operator with constant coefficients. Assume that $L=\sum_{k=0}^{p}A_k\partial^k/\partial t^k$. The initial value problem consists in finding a

$$u(x_1, \cdots, x_n, t)=u(x, t)$$

satisfying

$$Lu=m(x, t);\quad \lim_{t\to 0}\partial^k u/\partial t^k=\alpha_k(x)\quad \text{for}\quad k=0, \cdots, p-1.$$

The author formulates a corresponding problem for distributions in the sense of L. Schwartz. Using matrix notation, the distribution $\mathcal{T}$ shall satisfy the equation

$$L\mathcal{T}=T[m(x, t)H_t]+\left(1, \frac{\partial}{\partial t}, \cdots, \frac{\partial^{p-1}}{\partial t^{p-1}}\right)\times\begin{pmatrix}A_0 & A_1 & \cdots & A_{p-1}\\ 0 & A_0 & \cdots & A_{p-2}\\ \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & \cdots & A_0\end{pmatrix}\begin{pmatrix}T[\alpha_{p-1}(x)\delta_t]\\ T[\alpha_{p-2}(x)\delta_t]\\ \cdot\\ T[\alpha_0(x)\delta_t]\end{pmatrix}.$$

Here T denotes the distribution of a function, and H_t and δ_t are respectively the Heaviside and Dirac functions. It is shown that, if $\mathcal{T}$ is a solution, which is the distribution of a sufficiently regular function $u(x, t)$, then u is a solution of the initial value problem in the ordinary sense.

F. John (New York, N. Y.).

Garnir, H. G. Sur les distributions résolvantes des opérateurs de la physique mathématique. III. Bull. Soc. Roy. Sci. Liège **20**, 271–296 (1951).

Cet article utilise systématiquement la méthode de la transformation de Laplace, par rapport à la variable t seulement, des distributions dans l'espace-temps des x, t [exposée dans un article antérieur, même Bull. **20**, 174–194 (1951); ces Rev. **13**, 243]. Pour que $G_{x,t}$ soit distribution résolvante de l'opérateur différentiel à coefficients constants $L(\partial/\partial x, \partial/\partial t)$, il faut et il suffit que

$$G_{x,t}\left[\exp(-pt)L\left(-\frac{\partial}{\partial x}, p\right)\varphi(x)\right]=\varphi(0).$$

Cette méthode ramène les équations hyperboliques classiques à des équations elliptiques, résolues dans l'article cité. (Les résultats sont connus, mais la méthode exposée est rapide.) L'auteur donne ainsi les distributions résolvantes des opérateurs

$$\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\pm k^2,\quad \Delta-\frac{1}{k}\frac{\partial}{\partial t},\quad \left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)^2,$$

$$\Delta\Delta+\frac{1}{k^2}\frac{\partial^2}{\partial t^2},\quad \Delta\left(\Delta-\frac{1}{k}\frac{\partial}{\partial t}\right)$$

pour un espace à $n=1, 2, 3$, dimensions.

Note du reviewer. La distribution de Dirac δ, correctement définie et utilisée ($\delta(\varphi)=\varphi(0)$) est appelée par l'auteur "fonction de Dirac" égale à 0 partout et à 1 à l'origine. La distribution que l'auteur appelle $\delta(t-r/c)$ désigne la mesure portée par le demicône d'ondes, et de densité superficielle 1 (pour l'aire définie par la métrique $dx^2+dy^2+dz^2+dt^2$). Ce point de vue n'est pas correct dans la théorie du changement de variables en distributions. Aussi l'auteur est-il obligé d'inscrire un facteur $c/[1+c^2]^{\frac{1}{2}}$ (non homogène au point de vue physique) qui avec la définition correcte de $\delta(t-r/c)$ ne devrait pas figurer. *L. Schwartz* (Nancy).

Bers, Lipman. The expansion theorem for sigma-monogenic functions. Amer. J. Math. **72**, 705–712 (1950).

A Σ-monogenic function $f(z)$ is a complex-valued function $f=u+iv$ whose real and imaginary parts satisfy the partial differential equations

$$(*)\qquad \sigma_1(x)u_x=\tau_1(y)v_y,\quad \sigma_2(x)u_y=-\tau_2(y)v_x,$$

[L. Bers and A. Gelbart, Quart. Appl. Math. **1**, 168–188 (1943); Trans. Amer. Math. Soc. **56**, 67–93 (1944); these Rev. **5**, 25, 328; **6**, 86]. In this paper it is assumed that the functions $\sigma_1, \tau_1, \sigma_2, \tau_2$ are of class C'' and positive, and f is said to be Σ-monogenic only if u and v are of class C''. If u and v are of class C'' and satisfy instead of (*) the system

$$\frac{1}{\sigma_2(x)}\cdot u_x=\tau_1(y)v_y,\quad \frac{1}{\sigma_1(x)}\cdot u_y=-\tau_2(y)v_x,$$

then $f=u+iv$ is called Σ'-monogenic. The Σ-integral of a continuous function $f=u+iv$ is defined by the relation

$$\int f\,d_\Sigma z=\int u\sigma_2\,dx-v\tau_2\,dy+i\int(v/\sigma_1)\,dx+(u/\tau_1)\,dy.$$

It is path-independent if $f(z)$ is Σ-monogenic and is in this case a Σ'-monogenic function of the upper (variable) limit of integration. The Σ-derivative of a Σ-monogenic function $f(z)$ is defined by

$$d_\Sigma f/d_\Sigma z=f^{[1]}(z)=\sigma_1 u_x+iv_x/\sigma_2=\tau_1 v_y-iu_y/\tau_2.$$

The higher Σ-, Σ'-derivatives of the Σ-monogenic function $f(z)$, provided they exist, are defined by the recurrence formulae

$$f^{[0]}=f,\quad f^{[2n+1]}=d_\Sigma f^{[2n]}/d_\Sigma z,\quad f^{[2n]}=d_{\Sigma'}f^{[2n-1]}/d_{\Sigma'}z.$$

A significant class of Σ-monogenic and Σ'-monogenic functions is constituted by the "formal powers", which are defined (for any given complex numbers a and z_0) by the recurrence formulae:

$$a\cdot Z^{(0)}(z_0;z)=a\cdot\tilde{Z}^{(0)}(z_0;z)=a,$$

$$a\cdot Z^{(n)}(z_0;z)=n\cdot\int_{z_0}^{z}a\cdot\tilde{Z}^{(n-1)}(z_0;\zeta)\,d_{\Sigma'}\zeta,$$

$$a\cdot\tilde{Z}^{(n)}(z_0,z)=n\cdot\int_{z_0}^{z}a\cdot Z^{(n-1)}(z_0,\zeta)\,d_\Sigma\zeta.$$

If the sequence of complex numbers $a_0, a_1, a_2, \cdots$ satisfies $|a_\nu|^{1/\nu}=O(1)$ as $\nu\to\infty$, then the "formal power series"

$$(**)\qquad f(z)=\sum_{\nu=0}^{\infty}a_\nu\cdot Z^{(\nu)}(z_0;z)$$

converges uniformly and absolutely (together with all series obtained from it by Σ-, Σ'-differentiation) in some neighborhood G of z_0; thus $f(z)$ is Σ-monogenic in G, possesses Σ-, Σ'-derivatives of all orders, and

$$f^{[\nu]}(z_0)=\nu!a_\nu,\quad \nu=0, 1, 2, \cdots.$$

Now let $f(z)$ be a given Σ-monogenic function defined in the neighborhood of z_0. Can it be expanded in a formal power series of the form (**)? An affirmative answer was given by A. Gelbart and the author [see p. 83 of the second paper cited above] under the hypothesis that the coefficients $\sigma_1, \tau_1, \sigma_2, \tau_2$ are analytic functions. In this paper the expansion theorem is proved under the much weaker assumption that the coefficients $\sigma_1, \tau_1, \sigma_2, \tau_2$ are twice continuously differentiable. *J. B. Diaz* (College Park, Md.).

Bers, Lipman. Partial differential equations and generalized analytic functions. II. Proc. Nat. Acad. Sci. U. S. A. **37**, 42–47 (1951).

This note is an abstract of results to be published in full elsewhere, and contains a reformulation and extension of the theory outlined in the preceding note under the same title [same Proc. **36**, 130–136 (1950); these Rev. **12**, 173]. The new theory is based on a differentiation process (and a companion integration process) and is therefore a true generalization of classical complex variable theory. It applies to all elliptic linear homogeneous partial differential equations of second order (and to systems of first order equations) for unknown functions of two variables. The new theory contains as special cases and unifies the results of the previous note on self-adjoint equations, the earlier theory of sigma-monogenic functions [L. Bers and A. Gelbart, Quart. Appl. Math. **1**, 168–188 (1943); Trans. Amer. Math. Soc. **56**, 67–93 (1944); these Rev. **5**, 25, 328; **6**, 86], and the extension of the latter theory by Markuševič [as reported by I. G. Petrovskiĭ, Uspehi Matem. Nauk (N.S.) **1**, 44–70 (1946); these Rev. **10**, 301].

Let complex valued functions of the real variables (x, y) be written as functions of $z=x+iy$, without implying analyticity. Two complex-valued functions $F(z)$ and $G(z)$, having Hölderian first derivatives in a domain D_0 of the complex plane are said to form a generating pair in D_0 provided that $\Im(\bar{F}G)>0$. The generalization of function theory presented here is based on assigning to a generating pair (F, G) the part played in classical complex variable

theory by $(1, i)$. Any function $w(z)$, defined in a domain D, $\bar{D} \subset D_0$, may be represented in a unique way by

$$w(z) = \varphi(z)F(z) + \psi(z)G(z),$$

where φ and ψ are real valued (this is to be compared with the usual representation: $w(z) = \varphi(z) + i\psi(z)$). The differentiation process is defined as follows: if for some z_0 in D the limit

$$\dot{w}(z_0) = \lim_{z \to z_0} \frac{w(z) - \varphi(z_0)F(z) - \psi(z_0)G(z)}{z - z_0},$$

exists and is finite, then $w(z)$ is said to possess at z_0 the (F, G)-derivative $\dot{w} = d_{(F,G)}w/dz$. The existence of $\dot{w}$ in a domain has as far reaching consequences as the existence of the classical derivative. A function possessing an (F, G)-derivative at all points of D is called (F, G)-pseudo-analytic in D. The (F, G)-integral of a continuous function $W(z)$, taken over a rectifiable arc Γ leading from z_0 to z_1, is defined by

$$\int_\Gamma W d_{(F,G)}z = F(z_1)\Re \int_\Gamma \frac{\bar{G}W}{F\bar{G} - \bar{F}G}dz - G(z_1)\int_\Gamma \frac{\bar{F}W}{F\bar{G} - \bar{F}G}dz.$$

Theorem: A continuous function is (F, G)-integrable (i.e., its (F, G)-integral vanishes over all closed curves homologous to zero in D) if and only if it is an (F, G)-derivative. By definition, a generating pair (F_1, G_1) is called a successor of (F, G) if (F, G)-derivatives are (F_1, G_1)-pseudo-analytic, and (F_1, G_1)-pseudo-analytic functions are (F, G)-integrable. (F, G) is then called a predecessor of (F_1, G_1). It may be shown that any generating pair $(F, G) \equiv (F_0, G_0)$ may be imbedded in a generating sequence of pairs $\{(F_\nu, G_\nu)\}$, $\nu = 0, \pm 1, \pm 2, \cdots$, such that each $(F_{\nu+1}, G_{\nu+1})$ is a successor of (F_ν, G_ν). With respect to such a generating sequence, it may be shown that there are certain pseudo-analytic functions, called (global) formal powers, denoted by $Z_\nu^{(\nu)}(a, z_0; z)$, which "behave like" the usual powers $a(z - z_0)^r$, where r is a real rational number and a and z_0 are complex numbers. In terms of these formal powers, numerous results analogous to those of function theory, e.g. Taylor and Laurent expansions for single and multiple valued functions, Runge's approximation theorem, etc., can be established. The theory of pseudo-analytic functions bears the same relationship to the general theory of elliptic equations as the theory of analytic functions does to that of the Laplace equation. This follows essentially from the fact that a function $w(z) = \varphi(z)\,F(z) + \psi(z)\,G(z)$, possessing continuous first partial derivatives, is (F, G)-pseudo-analytic if and only if

$$\tfrac{1}{2}(w_x + iw_y) \equiv w_{\bar{z}} = aw + b\bar{w}$$

where

$$a = -(\bar{F}G_{\bar{z}} - \bar{F}_{\bar{z}}\bar{G})/(F\bar{G} - \bar{F}G), \quad b = (FG_{\bar{z}} - F_{\bar{z}}G)/(F\bar{G} - \bar{F}G),$$

that is, if and only if $F\varphi_{\bar{z}} + G\psi_{\bar{z}} = 0$.

The author wishes to correct the following misprint: Equation (12) should read $F\varphi_{\bar{z}} + G\psi_{\bar{z}} = 0$. *J. B. Diaz.*

Lammel, Ernst. Über eine zur Differentialgleichung

$$\left(a_0\frac{\partial^n}{\partial x^n} + a_1\frac{\partial^n}{\partial x^{n-1}\partial y} + \cdots + a_n\frac{\partial^n}{\partial y^n}\right)U(x, y) = 0$$

gehörige Funktionentheorie. I. Math. Ann. 122, 109–126 (1950).

Each solution $U(x, y)$ of $(\partial^2/\partial x^2 + \partial^2/\partial y^2)U(x, y) = 0$, which is regular in a neighborhood of $(0, 0)$, may be represented in the form $U(x, y) = f(x + iy) + g(x - iy)$ where f and g are regular in the same neighborhood of $(0, 0)$. Similarly, if $U(x, y)$ satisfies the equation $(\partial/\partial x^2 - \partial^2/\partial y^2)U(x, y) = 0$, then $U(x, y) = f(x + y) + g(x - y)$; and if

$$(\partial^4/\partial x^4 + 2\partial^4/\partial x^2\partial y^2 + \partial^4/\partial y^4)U(x, y) = 0,$$

then $U(x, y) = \varphi(x, y) + y\psi(x, y)$, where φ and ψ are harmonic functions. The author gives the following generalization of the preceding representation: If $U(x, y)$ is a solution of the equation of the title, regular in a neighborhood of $(0, 0)$, where a_j, $j = 1, 2, \cdots, n$ are real constants, $a_n \neq 0$, and the "characteristic" equation $a_0 + a_1\rho + \cdots + a_n\rho^n = 0$, has l distinct roots ρ_λ, $\lambda = 1, 2, \cdots, l$, the multiplicity of the root ρ_λ being ν_λ, so that $\nu_1 + \nu_2 + \cdots + \nu_l = n$, then

$$U(x, y) = \sum_{\lambda=1}^{l}\sum_{\nu=1}^{\nu_\lambda} y^{\nu-1} f_{\lambda\nu}(x + \rho_\lambda y),$$

where $f_{\lambda\nu}$ is a power series in $x + \rho_\lambda y$. The proof is carried out explicitly only for $n = 3$:

$$\left(a_0\frac{\partial^3}{\partial x^3} + a_1\frac{\partial^3}{\partial x^2\partial y} + a_2\frac{\partial^3}{\partial x\partial y^2} + a_3\frac{\partial^3}{\partial y^3}\right)U(x, y) = 0;$$

for the equation

$$(*) \qquad \left(k\frac{\partial^3}{\partial x^3} + l\frac{\partial^3}{\partial x^2\partial y} + \frac{\partial^3}{\partial y^3}\right)U(x, y) = 0,$$

which is obtained from the preceding one by a simple change of variable. The argument involves introducing a certain algebra of ordered triples $(u_n(x, y, z), v_n(x, y, z), w_n(x, y, z))$ of homogeneous polynomial solutions of equation (*), which is analogous to the algebra of ordered pairs $(\mathrm{Re}\, z^n, \mathrm{Im}\, z^n)$, where $z = x + iy$, for Laplace's equation. This leads to the consideration of ordered triples of functions $(u(x, y, z), v(x, y, z), w(x, y, z))$ which are analogous to ordinary analytic functions. The corresponding Cauchy-Riemann equations and Cauchy's independence of the path of a certain integral are obtained. The domain of convergence of a "power series" of the form:

$$(a_0, b_0, c_0) + \sum_{n=1}^{\infty}(a_n, b_n, c_n)(x, y, z)^n,$$

where multiplication is to be understood in the sense of the algebra of triples referred to above, is determined.

J. B. Diaz (College Park, Md.).

Integral Equations

Vainberg, M. M. On the variational theory of characteristic values of nonlinear integral equations. Doklady Akad. Nauk SSSR (N.S.) 80, 309–312 (1951). (Russian)

Let $K(x, y) \in L_2(B \times B)$ be a symmetric positive-semidefinite kernel with the eigenvalues $\mu_1 \geqq \mu_2 \geqq \cdots$, and corresponding orthonormal eigenfunctions $\varphi_1(x)$, $\varphi_2(x)$, $\cdots$, and such that the sums $\sum_{i=1}^{n}\mu_i\varphi_i(x)\varphi_i(y)$ converge in L_p $(p \geqq 2)$. Then, for every $\xi = (\xi_1, \xi_2, \cdots) \in l_2$, the function

$$\omega_\xi(x) = \sum_{k=1}^{\infty}\mu_k^{-1}\xi_k\varphi_k(x)$$

is in L_p. Let $g(u, x)$ $(-\infty < u < \infty, x \in B)$ be a real-valued function which is such that the operator $hu = g(u(x), x)$ is continuous on $L_p(B)$ to $L_q(B)$ $(p^{-1} + q^{-1} = 1)$ and $g(0, x) = 0$. By investigating the functional

$$f(\xi) = \int_B dy \int_0^{u_\xi(y)} g(u, y)du$$

and its extrema in the region $\|\xi\| \leqq C$ the author establishes the existence of an eigenvalue μ for which the equation $\mu u(x) = \int_a K(x, y) g(u(y), y) dy$ has a solution

$$u(x) = \sum_{k=1}^{\infty} \mu_k^{\frac{1}{2}} \eta_k \varphi_k(x)$$

with $0 < \|\eta\| \leqq C$. *M. Golomb* (Lafayette, Ind.).

Roberts, J. H., and Mann, W. R. On a certain nonlinear integral equation of the Volterra type. Pacific J. Math. 1, 431–445 (1951).

This is an extension of certain results of an earlier paper by Mann and Wolf [Quart. Appl. Math. 9, 163–184 (1951); these Rev. 13, 134]. A particular case of the integral equation mentioned in the title arises in a boundary value problem of heat transfer in which Newton's law of cooling has been replaced by the weaker condition that the net rate of heat exchange from a gas to a solid is some function $G[u(0, t)]$ of the surface temperature $u(0, t)$. Here $u(x, t)$ is the temperature of the solid at x at time t. It is pointed out that in every heat problem of physical significance the following conditions must be satisfied if the gas is kept at the constant temperature of 1°: (1) $G[1] = 0$, (2) $G[u]$ is continuous, (3) $G[u]$ is strictly increasing. In the earlier paper it was shown that the solution $u(x, t)$ could easily be constructed if the surface temperature $u(0, t)$ were known. It was also shown that the function $u(0, t)$ must satisfy the non-linear integral equation

$$u(0, t) = \int_0^t \frac{G[u(0, \tau)] d\tau}{\pi^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}}.$$

In the present paper it is shown that conditions (1), (2) and (3) are sufficient to imply that the solution of the integral equation is unique and strictly increasing. (The earlier paper established only the uniqueness and nondecreasing nature of the solution under the additional hypothesis that G satisfy a Lipshitz condition.) The proofs are carried out for an integral equation which is actually more general than the one stated above. *H. P. Thielman* (Ames, Ia.).

Charles, H. Sur l'équation de la chaleur dans la théorie de l'hérédité. Bull. Soc. Roy. Sci. Liège 20, 374–377 (1951).

The integrodifferential equation

$$U_t - U_{xx} = \int_0^t K(t-\tau) U_{xx}(x, \tau) d\tau$$

is solved under the boundary conditions $U(x, 0) = 0$, $U(0, t) = \delta(t)$, $U(l, t) = 0$, where $\delta(t)$ is the impulse function. The solution is completed in the special cases when $K(t) = 1$ and $K(t) = -ce^{-ct}$. *R. V. Churchill.*

Vasil'ev, V. V. Solution of linear generalized integro-differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 609–614 (1951). (Russian)

The author uses a method ascribed to Nekrasov [Trudy Central. Aero-Gidrodinam. Inst., no. 190, pp. 12–17 (1934)] to solve the equation

$$(1) \qquad L[z(x)] + \lambda \int_a^b P[x, z(y)] dy = 0,$$

where

$$L[z(x)] = \frac{d^n z}{dx^n} + a_1(x) \frac{d^{n-1} z}{dx^{n-1}} + \cdots + a_n(x) z,$$

$$P[x, z(y)] = K_0(x, y) z(y) + K_1(x, y) \frac{dz}{dy} + \cdots + K_m(x, y) \frac{d^m z}{dy^m}.$$

In the case where $m < n$ equation (1) is written as $L[z(x)] = F(x)$ and its general solution

$$z(x) = Z(x) + \int_a^b G(x, \eta) F(\eta) d\eta$$

leads to an integral equation equivalent to (1)

$$F(x) + \lambda \int_a^b P\left[x, \int_a^b G(y, \eta) F(\eta) d\eta\right] dy + \lambda \int_a^b P[x, Z(y)] dy = 0,$$

which is solved by Fredholm's methods. If $m \geqq n$ equation (1) is differentiated $m - n + 1$ times and the above method is applied with $z^{(m-n+1)}$ replacing z. *M. Golomb.*

Vasilache, Sergiu. On general solutions of integro-differential equations of the form:

$$\sum_0^n H_i(x) y^{(i)}(x) = f(x) + \lambda \int_a^x K(x, s) y(s) ds.$$

Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. 2, 495–501 (1950). (Romanian. Russian and French summaries)

This note points out the well known fact that the solution of the integro-differential equation mentioned in the title is equivalent to the solution of the integral equation

$$y(x) = F(x) + \lambda \int_a^x K_1(x, s) y(s) ds$$

where $F(x)$ and $K_1(x, s)$ are expressed in an obvious way in terms of an assumed known fundamental system of solutions of the homogeneous differential equation

$$\sum_0^n H_i(x) y^{(i)}(x) = 0$$

T. H. Hildebrandt (Ann Arbor, Mich.).

Davison, B. Influence of a black sphere and of a black cylinder upon the neutron density in an infinite non-capturing medium. Proc. Phys. Soc. Sect. A. 64, 881–902 (1951).

The content of this paper appeared earlier in report form [National Research Council of Canada. Division of Atomic Energy. Documents no. **MT-88** (N.R.C. 1549), **MT-93** (N.R.C. 1550) (1944); **MT-135** (N.R.C. 1554) (1945); these Rev. 9, 444, 445, 593].

Functional Analysis, Ergodic Theory

Klee, V. L., Jr. Convex sets in linear spaces. Duke Math. J. 18, 443–466 (1951).

The writer gives an account of various properties of convex sets in spaces with convex topologies in a very general setting. The topics include so-called core topologies, local convexity properties, maximal varieties, separation and support problems, validation of a conjecture of the reviewer, non-support points of convex sets, and some results on positively homogeneous subadditive functionals. In part the generality stems from what is now a recognized guiding principle amounting to the observation that in general "interior" point can be replaced by "internal" point in many statements. Specifically this is the justification for the introduction of the core topologies by the author. A typical interesting extension is the author's theorem that for a

linear space topologized so that ry and $y+rx$ are continuous, a closed connected subset F is convex if each point of F has a convex neighborhood in the relative topology.

D. G. Bourgin (Urbana, Ill.).

da Silva Dias, C. L. Concerning the continuity of analytic functionals. Bol. Soc. Mat. São Paulo **3**, no. 1–2 (1948), 21–30 (1951). (Portuguese)

This note is concerned with a proof that a functional which is linear and analytic in the sense of L. Fantappiè is continuous. It appears to the reviewer that there is nothing new in the paper. Continuity of linear analytic functionals is a direct consequence of Fantappiè's integral representation of such a functional. A proof not involving the integral formula was given by Haefeli and Pellegrino [Comment. Math. Helv. **21**, 225–246 (1948); these Rev. **9**, 515]. The present author shows essentially that a linear and continuous functional has an integral representation and is therefore analytic. Actually, it has been pointed out that all functionals (not merely the linear ones) analytic in the sense of Fantappiè are continuous [Sebastião e Silva, Portugaliae Math. **9**, 1–130 (1950), see p. 108–109; these Rev. **11**, 524].

A. E. Taylor (Los Angeles, Calif.).

Marinescu, G. The method of approximate equations in quasi-normed linear spaces. I. Acad. Repub. Pop. Române. Bul. Şti. A. **1**, 965–972 (1949). (Romanian. Russian and French summaries)

The author generalizes a method of Kantorovič [Uspehi Matem. Nauk. N.S. **3**, no. 6(28), 89–185 (1948), especially chap. II, §§1, 2; these Rev. **10**, 380] for approximating functional equations in Banach spaces to quasi-normed linear spaces (for a definition the reader is referred to an unavailable paper in Stud. Cerc. Mat.; however, the distinctive feature seems to be replacement of the triangle axiom by $\|x+y\| \leqq \mu(\|x\|+\|y\|)$ [cf. Hyers, Bull. Amer. Math. Soc. **51**, 1–21 (1945), especially p. 5 and further references cited there; these Rev. **6**, 275]). Let T be a bounded linear continuous operator defined on a linear quasi-normed space L, and let $\bar{T}$ be another (approximating) operator defined on a complete quasi-normed space $\bar{L}$. Let $\bar{L}=\varphi(L)$ where φ is linear and continuous and furthermore the restriction of φ to $L' \subset L$ is an isomorphism φ_0. (I) Suppose there exists $\epsilon>0$ such that $\|\varphi(T(x'))-\bar{T}(\varphi(x'))\| \leqq \epsilon\|x'\|$ for each $x' \in L'$ and (II) suppose there exists $\epsilon_1>0$ such that to each $x \in L$ corresponds $x' \in L'$ such that $\|T(x)-x'\| \leqq \epsilon_1\|x\|$. Let $S=I-\lambda T$, $\bar{S}=I-\lambda\bar{T}$. Theorem: If conditions I and II are satisfied, if S has an inverse, and if

$$\mu q = \mu\bar{\mu}|\lambda| \cdot \|\varphi_0^{-1}\| \cdot \|S^{-1}\|(\epsilon\mu+\epsilon\epsilon_1\mu|\lambda|+\epsilon_1\|\varphi\| \cdot \|S\|) < 1,$$

then the equation $\bar{S}\bar{x}=\bar{y}$ has a solution $\bar{x}$ for each $\bar{y}$ such that $\|\bar{x}\| \leqq M\mu/(1-\mu q)$, where $M=\mu\|S^{-1}\| \cdot \|\varphi_0^{-1}\| \cdot \|\varphi\|(1+\epsilon|\lambda|)$. The theorem and method of proof are almost identical except for the feature of the quasi-norm with Theorem 1 of Kantorovič's paper [loc. cit., p. 408]. Application is made to equations in the space l^p.

J. V. Wehausen.

Vinokurov, V. G. On quasi-complements in separable Banach spaces. Doklady Akad. Nauk SSSR (N.S.) **81**, 337–340 (1951). (Russian)

Murray [Trans. Amer. Math. Soc. **58**, 77–95 (1945); these Rev. **7**, 124] has shown that in a separable reflexive Banach space X every closed subspace P admits a quasi-complement; that is a closed subspace R such that $P \cap R=0$ and $P+R$ is dense in X. The reviewer [Bull. Amer. Math. Soc. **52**, 322–325 (1946); these Rev. **7**, 455] showed that the hypotheses of completeness and reflexivity are unnecessary. The present author generalizes Murray's result and partially generalizes that of the reviewer by showing that when X is separable and complete but not necessarily reflexive then given any closed subspace Q such that $P \cap Q=0$ one can choose the quasi-complement R of P so as to contain Q.

Actually one can show that completeness is unnecessary and that the author's result is an immediate consequence of that of the reviewer. Simply pass to the quotient space X/Q and apply the reviewer's result to the closure of the image of P. The inverse image in X of the resulting quasi-complement will be the desired quasi-complement of P.

G. W. Mackey (Cambridge, Mass.).

Pachale, Helmut. Zur Theorie der Elementfolgen in Banachschen Räumen. Math. Nachr. **3**, 127–141 (1950).

Given a sequence $\alpha=(\alpha_n)$ of complex numbers let $\rho(\alpha)=\inf \sigma$ $(0<\sigma, \sum|\alpha_n|^\sigma<\infty)$ if such σ exist, and $\rho(\alpha)=\infty$ otherwise. The author considers sequences (x_n), (x_n^*) in a Banach space B and its conjugate B^*. Let

$$\rho^{(1)}(x_n)=\sup \rho(x_n, x^*) \qquad (x^* \in B^*)$$

and set $\rho^{(2)}=\rho^{(1)}/(\rho^{(1)}-1)$. Typical results are: (1) If $\rho(x_n, x^*)<\infty$, then inequalities of the form $0<A \leqq |\alpha_n| \leqq B$ are equivalent to weak* convergence of $(\alpha_n x_n)$ to 0 plus $\rho(x_n, y^*)<\infty$ for some y^* in B^*. (2) Let $\rho(x_n, x^*)<\infty$ and let $\sum\alpha_n x_n$ converge. Then $\rho(\alpha)^{-1}+\rho^{(2)}(x_n)^{-1} \geqq 1$. Applications are made to expansions in biorthogonal systems.

W. F. Eberlein (Madison, Wis.).

Kakutani, Shizuo. A proof of Schauder's theorem. J. Math. Soc. Japan **3**, 228–231 (1951).

Let X be a Banach space, X^* the space conjugate to X, T a linear bounded operator of X into itself, and T^* the linear operator on X^* conjugate to T. A well known theorem by Schauder [Studia Math. **2**, 183–196 (1930)] states that T is completely continuous if and only if T^* is completely continuous. In the present note a new proof of this theorem is given based on the following observation: a special case of Arzelà's well known theorem concerning a set F of uniformly bounded equicontinuous functions defined on a totally bounded metric space X can be given a formulation in which the roles played by the sets X and F are symmetric.

E. H. Rothe (Ann Arbor, Mich.).

Yood, Bertram. Properties of linear transformations preserved under addition of a completely continuous transformation. Duke Math. J. **18**, 599–612 (1951).

Let X and Y be two not necessarily different Banach spaces, and T a bounded linear map of X into Y. The linear closed manifold in Y determined by the image set $T(X)$ is denoted by $N(T)$. If $T(X)$ is closed, i.e. if $T(X)=N(T)$, T is called a homomorphism. The nullset of T, i.e. the set of $x \in X$ annihilating T, is denoted by $M(T)$, and its dimension by m. A homomorphism T is said to have property A if m is finite, and property B if the dimension n of the quotient space $Y/N(T)$ is finite. If T has both properties the number $j=n-m$ is called the index of T. Mappings of the properties A or B had been investigated by Dieudonné [Bull. Sci. Math. (2) **67**, 72–84 (1943); these Rev. **7**, 124], and the main part of the present paper deals with further properties and with the characterization of such maps and their iterates. As an example of the results obtained we mention the following theorem: T has property A (property B) either without property B (property A) or with index $\geqq 0$ $(\leqq 0)$ if and only if $T=U_1+U_2$ where the linear bounded

map U_1 of X into Y is an isomorphism (is a map onto Y) and where U_2 is linear and completely continuous. Some of the theorems obtained contain as special case classical results of Riesz and Schauder concerning maps $X \to X$ of the form $T = I + U$ where I is the identity map and U linear and completely continuous. *E. H. Rothe.*

Nemyckiĭ, V. V. Some questions of the structure of the spectrum of nonlinear completely continuous operators. Doklady Akad. Nauk SSSR (N.S.) **80**, 161–163 (1951). (Russian)

Let A be an operator transforming a real Banach space into itself and leaving the zero element θ fixed. The spectrum of A is the set $S(A) = \{\lambda \mid A\varphi = \lambda\varphi, \varphi \neq \theta\}$. Among various results concerning the structure of $S(A)$ stated in this paper are the following: If A is completely continuous, then $S(A)$ is of type F_σ. If

$$\lim_{\|\varphi\| \to 0} \|A\varphi\|/\|\varphi\| = 0 \quad \text{and} \quad \lim_{\|\varphi\| \to \infty} \|A\varphi\|/\|\varphi\| = \infty,$$

then $S(A)$ is closed. $S(A)$ contains interior points if A satisfies certain conditions, and the interior point λ_0 corresponding to the eigenelement φ_0 is "stable" if the transformation $\psi = (A - \lambda_0)\varphi$ is one-to-one and uniformly continuous in some neighborhood of φ_0. *M. Golomb.*

Shimoda, Isae. On the behaviour of power series on the boundary of the sphere of analyticity in abstract spaces. J. Sci. Gakugei Fac. Tokushima Univ. **1**, 1–5 (1950).

Let $f(x) = \sum_{n=0}^{\infty} h_n(x)$ be a power series on E to E' (complex Banach spaces); $h_n(x)$ is a homogeneous polynomial of degree n. The radius of analyticity τ of the series is the largest number ρ such that the series converges and defines a function $f(x)$ analytic when $\|x\| < \rho$. In an earlier paper [Shimoda, Math. Japonicae **1**, 69–73 (1948); these Rev. **10**, 307] it was shown that

$$\frac{1}{\tau} = \sup_{G \in K} \left\{ \limsup_{n \to \infty} [m_n(G)]^{1/n} \right\},$$

where K is the family of all compact subsets of the surface of the unit sphere in E, and $m_n(G) = \sup_{x \in G} \|h_n(x)\|$. In the present paper the following result is established: In order that the power series shall converge absolutely and uniformly in some neighborhood of each point on the boundary of the sphere $\|x\| = \tau$, it is necessary and sufficient that $\limsup_{n \to \infty} [m_n(G)]^{1/n} < 1/\tau$ for each $G \in K$. An example is given in which this situation actually occurs. Of course E must be infinite-dimensional for such an example. The foregoing formulation of the principal theorem is not quite the same as the formulation given by Shimoda. There is some confusion caused by use in the paper of the two statements "the power series is analytic at the point y on the sphere $\|x\| = \tau$" and "the power series has a singular point at y on the sphere $\|x\| = \tau$" without explicit definition of what the statements mean. The reviewer finds that the theorems in the paper are true if the first statement means that the power series converges uniformly in some neighborhood of y, and if the second statement is the negation of the first. Neither of these meanings is in accord with common usage. With the usual meaning for a singular point (namely, that the function defined by the series has no analytic continuation across the sphere into a full neighborhood of the point), it would be interesting to know if the absence of singularities on the surface of the sphere will guarantee the convergence of the power series in some neighborhood of each point of the boundary of the sphere. The reviewer conjectures that there is no such guarantee. *A. E. Taylor.*

James, Robert C. Linear functionals as differentials of a norm. Math. Mag. **24**, 237–244 (1951).

This is an expository paper. It deals with the Gâteaux differential of the norm functional on a Banach space and the representability of linear functionals as scalar multiples of this Gâteaux differential. It is based mainly on the author's work [Trans. Amer. Math. Soc. **61**, 265–292 (1947); these Rev. **9**, 42]. *B. Yood* (Ithaca, N. Y.).

James, Robert C. A non-reflexive Banach space isometric with its second conjugate space. Proc. Nat. Acad. Sci. U. S. A. **37**, 174–177 (1951).

The author has previously constructed a separable, non-reflexive Banach space B which is isomorphic with B^{**} [Ann. of Math. (2) **52**, 518–527 (1950); these Rev. **12**, 616]. By modifying the norm of B he now constructs a space E isomorphic to B and such that E is isometric with E^{**}. *W. F. Eberlein* (Madison, Wis.).

Michal, Aristotle D. Invariant differential forms in several group variables as solutions of partial differential equations in Fréchet differentials. Proc. Nat. Acad. Sci. U. S. A. **37**, 766–771 (1951).

Let B and B_1 be real Banach spaces, and let Δ be both an open set in B and a topological multiplicative group of class C_2 (all differentials in the sense of Fréchet). A connectivity condition is placed on Δ. The first and second parameter groups are π_1: $\tilde{\alpha} = \alpha\beta$, with parameter β, and π_2: $\tilde{\beta} = \alpha\beta$, with parameter α. Let $W(\beta_1, \beta_2, \delta\beta_1, \delta_1\beta_2, \delta_2\beta_2)$ be a function with values in B_1, defined for $\beta_1, \beta_2 \in \Delta$, $\delta\beta_1, \delta_1\beta_2, \delta_2\beta_2 \in B$, of class C_1, linear in each of the last three variables, and alternating (skew-symmetric) in the last two. It is announced by the author that, as a necessary and sufficient condition for invariance of the two-point differential form W under the group π_2, W must satisfy a certain partial differential equation. Only a few remarks are given as to method of proof. Two other theorems, dealing with simultaneous invariance under π_1 and π_2, are stated. *A. E. Taylor.*

Wilansky, Albert. The basis in Banach space. Duke Math. J. **18**, 795–798 (1951).

Let $\{x_n; f_n\}$ be a basis for a Banach space B. Lemma: If $Y \in B^{**}$, then $\sup_n \|\sum_1^n Y(f_k)x_k\| < \infty$. If $\sup_n \|\sum_1^n a_k x_k\| < \infty$, then there is a $Y \in B^{**}$ with $Y(f_k) = a_k$. If $h \in B^*$, then $\sup_n \|\sum_1^n h(x_k) f_k\| < \infty$. If $\sup_n \|\sum_1^n a_k f_k\| < \infty$, there is an $h \in B^*$ with $h(x_k) = a_k$. This lemma is used to show that the following properties are equivalent. (1) $\{f_n; x_n\}$ is a basis for B^*. (2) $\|h\|_n$ = norm of h_n on the closed linear hull of $x_n, x_{n+1}, \cdots \to 0$. (3) $\lim_n f_k(y_n) = 0$ and y_n bounded imply y_n converges weakly to 0. (4) $f_k(y_n) = 0$ for $n > k$ and y_n bounded imply y_n converges weakly to 0. The conjunction of any one with (5) $\sup_n \|\sum_1^n a_k x_k\| < \infty$ implies $\sum_1^\infty a_k x_k$ converges, is equivalent to the reflexivity of B. (5) holds for B^* if and only if (1) holds for B. A result of James [Ann. of Math. (2) **52**, 518–527 (1950); these Rev. **12**, 616] is more simply proved by avoiding the use of Eberlein's result on reflexivity and weak compactness [Proc. Nat. Acad. Sci. U. S. A. **33**, 51–53 (1947); these Rev. **9**, 42; **10**, 855]. A statement concerning the strengthening of Theorem 6 of the reviewer [Duke Math. J. **17**, 187–196 (1950); these Rev. **11**, 729] is false in that property (1) above is certainly equivalent to the existence of a retro-basis. Furthermore, Eberlein's reflexivity theorem for spaces with bases is proven more simply and the proof of Theorem 10 of Eberlein's paper is shortened. *B. Gelbaum* (Minneapolis, Minn.).

Wilansky, Albert. Norms of matrix type for the spaces of convergent and bounded sequences. Proc. Amer. Math. Soc. 2, 738–741 (1951).

If $x=\{x_n\}$ is a sequence of complex numbers, we may define $\|x\|_A=\sup_n|\sum_{k=1}^{\infty}a_{nk}x_k|=\sup_n|A_n(x)|$, where $A=(a_{nk})$ is an infinite matrix. The author considers necessary conditions and sufficient conditions for the space (c) of convergent sequences and the space (m) of bounded sequences to be Banach spaces under the above norm. He proves, inter alia, that if A is normal and conservative, and (m) is a Banach space, so is (c). *A. F. Ruston* (London).

Krasnosel'skiĭ, M. A., and Rutickiĭ, Ya. B. On the theory of Orlicz spaces. Doklady Akad. Nauk SSSR (N.S.) 81, 497–500 (1951). (Russian)

Because of their relevance to the study of non-linear operators, the Orlicz spaces [W. Orlicz, Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Sér. A. Sci. Math. 1932, 207–220] are discussed. They are the spaces L^*_Φ of measurable functions $u(x)$, $x\in C$, C a compact subset of Euclidean space, for which $|u|_\Phi=\sup|\int_C u(x)v(x)dx|<\infty$, sup over $v(x)$ for which $\int_C\Psi(v(x))dx<\infty$. Here Ψ is the conjugate of a function Φ, where Φ is a convex function satisfying: $\Phi(0)=0$, $\Phi(-u)=\Phi(u)>0$ for $u\neq 0$, $\lim_{|u|\to 0}\Phi(u)/|u|=0$, $\lim_{|u|\to\infty}\Phi(u)/|u|=\infty$. It is known that $\Phi(u)=\int_0^{|u|}p(t)dt$ where $p(t)$ is non-decreasing and continuous on the right, and such that $p(t)>0$ for $t>0$, $\lim_{t\to 0}p(t)=0$, $\lim_{t\to\infty}p(t)=\infty$. If $q(s)=\sup_{p(t)\leq s}t$, and $\Psi(v)=\int_0^{|v|}q(s)ds$, then Ψ is the conjugate of Φ. If $\limsup\Phi(2u)/\Phi(u)=k<\infty$ (Δ_2-condition), certain classes of non-linear operators are no more easily investigated in L^*_Φ than in L_p for any p. The following theorem is a criterion for the Δ_2-condition. The conjugate Ψ of Φ satisfies the Δ_2-condition if and only if there is a u_0 and a constant $l>1$ for which $\Phi(u)\leqq\Phi(lu)/2l$, $u\geqq u_0$. An example of a pair Φ, Ψ neither of which satisfies the Δ_2-condition is given. If Φ does not satisfy the Δ_2-condition then the bounded functions are nowhere dense in L^*_Φ, $L^*_\Phi\subset L^*_{\Phi_1}$, if and only if for some constant s and some u_0, $\Phi(u)\leqq\Phi_1(su)$, $u\geqq u_0$. The $u_n(x)$ are said to converge weakly whenever $\int_C u_n v dx$ converges for all v in L^*_Ψ. Since L^*_Ψ is in general a proper subset of the conjugate of L^*_Φ, it follows that this is a non-usual weak convergence. If $u_n(x)$ converge weakly, $|u_n|_\Phi$ are bounded. If Ψ satisfies the Δ_2-condition, then every bounded sequence has a weakly convergent subsequence and every weakly converging sequence has a weak limit. *B. Gelbaum* (Minneapolis, Minn.).

Haplanov, M. G. A matrix criterion for a basis in a space of analytic functions. Doklady Akad. Nauk SSSR (N.S.) 80, 177–180 (1951). (Russian)

A sequence $\{f_n(z)\}$, $n=0,1,\cdots$, of functions analytic in a region B containing $z=0$ is called a "quasi-power basis" in B if there exists $R_1>0$ such that $\sum c_n f_n$ converges uniformly in B for every sequence c_n satisfying $\limsup|c_n|^{1/n}\leqq 1/R_1$ and if every function analytic in B has such an expansion. The matrix M of the basis is the infinite matrix whose $(k+1)$st column consists of the coefficients of the MacLaurin series expansion of f_k. The author proves: $\{f_n(z)\}$ is a quasi-power basis in $|z|<R$ if and only if M transforms some analytic space A_{R_1} into A_R and has a unique inverse transforming A_R into A_{R_1} [cf. Haplanov, same Doklady (N.S.) 79, 929–932 (1951); these Rev. 13, 252]. Several methods are indicated for obtaining bases in other regions from a given one in $|z|<R$. *B. Crabtree* (Durham, N. H.).

Magnus, Wilhelm. Über einige beschränkte Matrizen. Arch. Math. 2 (1949–1950), 405–412 (1951).

The two infinite matrices

$$H(\theta)=\left(\frac{1}{n+m+\theta}\right) \quad \text{and} \quad A(\theta)=\frac{\sin\pi\theta}{\pi}\left(\frac{1}{-n+m+\theta}\right)$$

are investigated, where θ is a real parameter. They are known to be bounded. Among the many results obtained is the fact that $I-[\pi^{-1}\sin\pi\theta H(1+\theta)]^2$ has two left and right inverses. Being bounded, it can have only one bounded left and right inverse which is given by the power series $\sum[\pi^{-1}\sin\pi\theta H(1+\theta)]^{2n}$ for $-\frac{1}{2}<\theta<\frac{1}{2}$ and $\frac{1}{2}<\theta<1$. Since the other inverse mentioned is bounded for $-\frac{1}{2}<\theta<\frac{1}{2}$ it coincides there with the power series, but not for $\frac{1}{2}<\theta<1$ since it is unbounded in the latter range. Generalizations of Hilbert's inequality are obtained. Further a matrix $S(\theta)$ is found such that $S(\theta)S(\theta)$ and $S(\theta)[S(\theta)S(\theta)]$ exist while $[S(\theta)S(\theta)]S(\theta)$ does not exist. Further, the inhomogeneous system of linear equations $A(\theta)x=y$ is studied [with this compare Titchmarsh, Proc. London. Math. Soc. (2) 26, 1–11 (1926); Linfoot and Shepherd, Quart. J. Math., Oxford Ser. 10, 84–98 (1939); Hilding, ibid. 17, 240–244 (1946); these Rev. 8, 375]. Results concerning the spectrum of $H(\theta)$ are obtained [see also Magnus, Amer. J. Math. 72, 699–704 (1950); these Rev. 12, 836]. *O. Taussky-Todd.*

Day, Mahlon M. Means for the bounded functions and ergodicity of the bounded representations of semi-groups. Trans. Amer. Math. Soc. 69, 276–291 (1950).

Given an abstract semi-group Σ let $m(\Sigma)$ denote the Banach space of bounded real-valued functions on Σ. The author shows first that weak ergodicity in the sense of the reviewer [Trans. Amer. Math. Soc. 67, 217–240 (1949); these Rev. 12, 112] of every bounded representation of Σ as operators on a Banach space is equivalent to the weak ergodicity of the right and left regular representations on $m(\Sigma)$, in turn equivalent to the existence of a two-sided invariant mean on $m(\Sigma)$. Uniform ergodicity is related to the existence of finite means converging in the norm of $m(\Sigma)$ to two-sided invariance.

Sufficient conditions for existence of invariant means and finite means converging to invariance are investigated. The known result that a bounded Abelian semi-group of operators is (uniformly) ergodic under its finite means is extended to the case of solvable groups and semi-groups. If G is a normal subgroup of a group H and if G and H/G have two-sided invariant means, so has H. If G has an invariant mean, every bounded representation of G on a Hilbert space is equivalent to a unitary representation. This last result, in particular, was obtained independently by Dixmier [Acta Sci. Math. Szeged 12, Pars A, 213–227 (1950); these Rev. 12, 267]. *W. F. Eberlein* (Madison, Wis.).

Nicolescu, Miron. On the criterion of compactness of A. Kolmogorov. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. 2, 407–415 (1950). (Romanian. Russian and French summaries)

For $x\in E^n$ and $\epsilon>0$ let $S(x,\epsilon)$ be the sphere with center x and radius ϵ, $V(\epsilon)$ the volume of $S(x,\epsilon)$. For a function f on E^n, let $f_\epsilon(x)=V(\epsilon)^{-1}\int_{S(x,\epsilon)}f$. Kolmogorov [Nachr. Ges. Wiss. Göttingen 1931, 60–63] established for subsets F of L^p (over E^n) a compactness criterion involving mean approximability of f by f_ϵ (uniformly for $f\in F$). By using Kolmogorov's result and a simple diagonal process, the author strengthens the former and obtains a short proof of an extension by Ta-

markin [Bull. Amer. Math. Soc. 38, 79–84 (1932)]. [Reviewer's notes: (1) On p. 409, line 16, "mărginită" should be inserted between "mulțime" and "a". (2) Kolmogorov's proof applies only for $p>1$. However, Tulajkov [Nachr. Ges. Wiss. Göttingen 1933, 167–170] (not in the author's bibliography) established the Kolmogorov-Tamarkin criterion for all $p\geqq 1$, and the author's method can be applied advantageously here also.]

The author proves also that if F is a uniformly bounded set of functions on E^n such that $\lim_{x\to 0}|f_x(x)-f(x)|=0$ uniformly for $f\in F$ and $x\in E^n$, then F is normal (i.e., compact in the topology of uniform convergence). Several applications of this result are given, of which we mention two: (A) If Φ is continuous, Δ is the generalized Laplacian of Privaloff [C. R. (Doklady) Acad. Sci. URSS (N.S.) 31, 102–103 (1941); these Rev. 2, 366] and U is a uniformly bounded set of integrals of the equation, $\Delta u=\Phi(x, u)$, then U is normal. (B) If U is a uniformly bounded set of functions on an interval $[a, b]$ such that

$$\lim_{h\to 0}|u(x+h)-2u(x)+u(x-h)|=0$$

uniformly for $x\in[a, b]$ and $u\in U$, then U is normal.

V. L. Klee, Jr. (Princeton, N. J.).

✓ ***Aronszajn, N. Introduction to the Theory of Hilbert Spaces. Vol. I.** Mathematical Monographs. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1950. iv+147 pp.

This volume is the first of a series intended to serve as an introduction to Hilbert spaces and its applications. It is written in a clear style and presents the elementary notions of the theory in a coordinate-free form and in a half analytic, half geometric way. The main headings of the various sections refer to vector spaces, positive forms, Cauchy-Schwarz and Minkowski inequalities, Banach and Hilbert spaces, completion, orthogonality, functionals and linear transformations, weak topology, etc. It is said in the preface that subsequent chapters will include the study of linear transformations and the theory of spectral decomposition. The booklet should prove helpful to the students of the subject, it being pertinent to point out here that the weak topology is presented by the author mostly in its sequential aspect and only occasionally mentioned in terms of neighborhoods.

L. Nachbin (Rio de Janeiro).

***Ahiezer, N. I., i Glazman, I. M. Teoriya linelnyh operatorov v gil'bertovom prostranstve. [The Theory of Linear Operators in Hilbert Space].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 483 pp.

The first six chapters of this book present a clear and detailed exposition of the standard theory of linear operators in Hilbert space, culminating in the spectral theorems for self-adjoint and unitary operators. The seventh chapter and two appendices are concerned with the extension of symmetric operators and with the theory of differential operators. This material is less elementary and well known than that of the first six chapters and some of it represents quite recent work.

The chapter and appendix headings are as follows: Chapter I, Hilbert space; Chapter II, Linear functionals and bounded linear operators; Chapter III, Projections and unitary operators; Chapter IV, Certain general ideas and propositions in the theory of linear operators; Chapter V, Spectral analysis of completely continuous operators; Chapter VI, Spectral analysis of unitary and self adjoint operators; Chapter VII, Theory of extension of symmetric linear operators; Appendix I, Generalized extensions and generalized spectral functions of symmetric operators; Appendix II, Differential operators.

Chapter I begins with the introduction of the notions of vector space, scalar product, metric space, etc., leading up to the definition of abstract Hilbert space. There follows a treatment of orthogonal projection and complete orthonormal systems and the chapter concludes with a number of examples. The main topics treated in chapter II are the Riesz representation theorem for linear functionals, the form of the general bilinear functional, adjoints of operators, the weak topology, the notion of complete continuity and the various senses in which a sequence of operators may converge.

In chapter III the elementary algebra of projections is developed, unitary and isometric operators are defined and briefly discussed and the Fourier-Plancherel operator is analyzed as an example. In chapter IV unbounded operators are introduced and the notions of closedness and symmetry for these operators are developed, making use of the von Neumann graph of an operator. The elementary facts about spectra and resolvents are also presented here and the chapter concludes with a detailed discussion of certain classes of examples; operators defined by matrices, differential and integral operators, and multiplication operators. In chapter V the spectral theory of completely continuous operators is developed, first in general and then in more detail for self adjoint and normal operators in Hilbert space. In a final section the results are applied to deduce the fundamental theorem in the theory of almost periodic functions on the line. The sixth chapter begins with a number of theorems about the representation of functions as Stieltjes integrals and these are then used to establish the various spectral theorems, first for single unitary operators, next for one parameter groups of unitary operators, then for resolvents of self-adjoint operators, and finally for single self-adjoint operators. Topics discussed in the second half of this rather long (117 page) chapter include the Cayley transform, spectral multiplicity theory, and functions of operators. The discussion of spectral multiplicity theory includes a proof of the unitary equivalence theorem for operators with simple spectra but does not attempt to deal with the general case.

A part of the seventh chapter is devoted to the classical material on the deficiency index and the structure of maximal symmetric operators. It includes in addition some recent work of Krein on the resolvents of the self-adjoint extensions of a symmetric operator and a treatment of the relationship between the spectrum of a symmetric operator and those of its self-adjoint extensions. The generalized extensions referred to in the title of appendix I are operators in a Hilbert space properly including the original one. The appendix begins with Neumark's theorem on the connection between a generalized (the $E(t)$ are non-negative self-adjoint operators which are not necessarily projections) resolution of the identity and an ordinary resolution of the identity in a larger Hilbert space. Other topics treated are generalized resolvents and some recent work of Lifschitz on "quasi-self-adjoint extensions". The differential operators considered in appendix II are of order $2n$ and operate on functions of one real variable. The theory is essentially that developed by Stone for the case $n=1$ and extended to the general case by Shin. The treatment given here includes some contributions due to the second named author of the book under review and appearing in his 1949 Kharkov

thesis. Considerable space is devoted to eigen-function expansions. *G. W. Mackey* (Cambridge, Mass.).

***Hamburger, Hans. On a new characterization of self-adjoint differential operators in the Hilbert space $\mathfrak{L}_2$.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 229–247. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

The purpose of the present paper is to continue certain previous research of the author [cf. Ann. of Math. (2) **47**, 667–687 (1946); these Rev. **8**, 276] on the question of when a symmetric operator H of deficiency (m, m) on a space $\mathfrak{L}_2(\Delta)$, Δ a finite or infinite interval, can be represented as a formally self-adjoint differential operator or *SL* operator. The author has defined a set of operators "of class P_*" which can be described heuristically as having the property that values of Hf regarded as a function in $\mathfrak{L}_2$ on any interval, no matter how small, depend only on the values of f in this interval. Operators of class P_* which satisfy two further conditions one of which involves the domain of H have been shown to be *SL* operators [cf. loc. cit.]. In the present paper a Volterra form for the inverse of operators of class P_* is obtained. From this form, a subset of these operators is shown to correspond to differential operators having singularities and consequently the operators of class P_* are definitely larger than the class of *SL* operators for every m. *F. J. Murray* (New York, N. Y.).

Halmos, Paul R. Normal dilations and extensions of operators. Summa Brasil. Math. **2**, 125–134 (1950).

Let the Hilbert space H be a subspace of the Hilbert space K, and let $H^\perp$ be the orthogonal complement of H in K. Since any element $z \in K$ can be expressed uniquely as $z=x+y$, where $x \in H$, $y \in H^\perp$, we may represent any bounded linear operator B from K into itself by the matrix $\begin{pmatrix} Q & S \\ T & R \end{pmatrix}$, where Q, S, T and R are uniquely determined bounded linear operators from H into H, $H^\perp$ into H, H into $H^\perp$ and $H^\perp$ into $H^\perp$ respectively, and $Bz=(Qx+Sy)+(Tx+Ry)$. In order that B be an extension of the bounded linear operator A from H into itself (or that A be a restriction of B), it is necessary and sufficient that $T=0$ and $Q=A$. The author considers also a weaker relation between A and B. If $Q=A$, he says that B is a dilation of A to K and that A is a compression of B in H.

The author proves, inter alia, that every bounded linear operator A on every Hilbert space H has a normal dilation. A consequence of this is that any two Hermitian operators have commutative dilations. He also deduces that the weak closure of the set of all projections on an infinite dimensional Hilbert space is the set of all positive linear operators of norm not greater than 1. In order to deal with extensions, he introduces two properties, namely complete subnormality and complete boundedness, which together are necessary and sufficient for a given bounded linear operator to have a normal extension. *A. F. Ruston* (London).

Yosida, Kôsaku. On the unitary equivalence in general Euclid space. Proc. Japan Acad. **22**, no. 8, 242–245 (1946).

Let T_1, T_2 be self-adjoint operators in a Euclid, that is, unitary, space R of arbitrary dimension. Let $(T_1)'$, $(T_2)'$ be the rings of bounded linear operators which commute with T_1, T_2 respectively. The author proves that for unitary equivalence of T_1 and T_2 it is sufficient that $(T_1)'$ and $(T_2)'$ be isomorphic by a correspondence C which maps adjoints onto adjoints and maps T_1 onto T_2. (It is obvious that this condition is necessary.) It is first shown that C must map $E_1(\lambda)$ onto $E_2(\lambda)$ with $T_1=\int \lambda dE_1(\lambda)$, $T_2=\int \lambda dE_2(\lambda)$. Now for elements x, y in R let $M_1(x)$, $M_2(y)$ denote the closed linear separable manifolds spanned by the $E_1(\lambda)x$, $E_2(\lambda)y$ respectively and let $E_1(x)$, $E_2(y)$ denote projections onto $M_1(x)$, $M_2(y)$ respectively. Then it is shown that every $E_1(x)$ must be mapped by C onto some $E_2(y)$ and that a unitary mapping of $M_1(x)$ onto $M_2(y)$ can be constructed which maps $E_1(\lambda)x$ onto $E_2(\lambda)y$ for every λ and hence maps $E_1(x)T_1$ onto $E_2(y)T_2$. The theorem follows by using a decomposition of R into mutually orthogonal such $E_1(x)$. *I. Halperin.*

***Dunford, Nelson. The reduction problem in spectral theory.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 115–122. Amer. Math. Soc., Providence, R. I., 1952.

Let $\mathfrak{X}$ be a complex Banach space, and T a bounded linear operator mapping $\mathfrak{X}$ into itself. The author poses the problem of finding a sufficiently large Boolean algebra $\mathfrak{B}$ of sets τ in the complex plane, and a homomorphic map E_τ of $\mathfrak{B}$ into a Boolean algebra of projections such that $E_\tau T=TE_\tau$ and such that the spectrum of T, as an operator on the subspace $E_\tau\mathfrak{X}$ is contained in the closure of τ. As a second part of the problem it is desired to express the integral

$$f(T)=\frac{1}{2\pi i}\int_C (\lambda I-T)^{-1}f(\lambda)d\lambda$$

in terms of the projections E_τ, the operators T^n $(n=0,1,2,\cdots)$, and values of f and its derivatives on the spectrum of T (denoted by $\sigma(T)$). Here f is any complex function, single-valued and analytic on a neighborhood of $\sigma(T)$, and C is the boundary of an appropriate region containing $\sigma(T)$ in its interior.

If T possesses a countably additive resolution of the identity E_τ defined for τ in the family $\mathfrak{B}$ of Borel sets, T is called a spectral operator. For such operators one has

$$f(T)=\sum_{n=0}^{\infty}\int_{\sigma(T)}\frac{f^{(n)}(\lambda)}{n!}(T-\lambda I)^n dE_\lambda,$$

the integral being of Riemann type and all convergence being in the uniform topology of operators. The results which follow are concerned with conditions which are necessary, or, under additional restrictions, necessary and sufficient for T to be a spectral operator.

For $x \in \mathfrak{X}$ let $x(\xi)$ be the maximal analytic extension of $(\xi I-T)^{-1}x$ and let $\rho(x)$ be the open set on which $x(\xi)$ is defined. Let $\sigma(x)$ be the complement of $\rho(x)$ and, for any set σ, let $[\sigma]=\{x \mid \sigma(x) \subset \sigma\}$. Suppose $x^* \in \mathfrak{X}^*$, and let σ be a set of singular points of $x^*x(\xi)$. Let $(x^*, x)_\sigma$ denote the residue (defined by an integral) of $x^*x(\xi)$ at the points of σ. Then the following three conditions are necessary for T to be a spectral operator. N(1): $x(\xi)$ is single-valued; N(2): $[\sigma]$ is closed if σ is closed; N(3): there is a constant K such that $|(x^*, x)_\sigma| \leq K|x| \cdot |x^*|$. If $\sigma(T)$ is nowhere dense and $\mathfrak{X}$ is weakly complete, conditions N(2) and N(3) are sufficient for T to be a spectral operator. Condition N(1) is automatically satisfied in this case.

For further work, attention is limited to operators for which $\sigma(T)$ lies on a Jordan curve Γ of class C''. The growth of the resolvent of T may be subjected to the following condition S(1): to $\zeta \in \Gamma$ corresponds an integer $\nu(\zeta)$ and a constant $K(\zeta)$ such that

$$|(\zeta-\xi)^{\nu(\zeta)}(\xi I-T)^{-1}| \leq K(\zeta)$$

when ξ is near ζ and on the normal to Γ at ζ. One can then show that, if $\mathfrak{X}$ is reflexive and S(1) is satisfied, T is a spectral operator if and only if N(3) is satisfied. A number of other theorems are stated, some involving spaces which are weakly complete but not necessarily reflexive, and some involving a mean rate of growth condition. Proofs are not given in the paper. *A. E. Taylor* (Los Angeles, Calif.).

* **Dunford, Nelson. Spectral theory in abstract spaces and Banach algebras.** Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 1–65. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

This is an exposition of the spectral theory of commutative B^*-algebras of operators in Hilbert space [for the definition of a B^*-algebra see, e.g., M. A. Naĭmark, Uspehi Matem. Nauk (N.S.) **3**, no. 5(27), 52–145 (1948) = Amer. Math. Soc. Translations no. **25** (1950), p. 24; these Rev. **10**, 308; **12**, 111; a B^*-algebra of operators is a B^*-algebra whose elements are bounded operators in some Hilbert space such that the operator and ring norms coincide and such that the *-operation coincides with the formation of the adjoint operator]. The presentation includes a theorem on the resolution of the identity associated with the algebra (Theorem 8.1), and on the operational calculus (Theorems 8.1 and 8.2); applications are given to normal operators (Theorems 8.3 and 8.4), Fourier and Dirichlet transforms (Chapter IX), and convolution operators in $L_2(-\infty, \infty)$ (Chapter X).

The approach is via the Gelfand-Naĭmark representation theorem of a commutative B^*-algebra as the ring of all continuous functions on a compact Hausdorff space and the Riesz representation theorem of bounded functionals on the space of continuous functions over a compact Hausdorff space by means of integrals with respect to regular measures. [K. Yosida [see K. Yosida and T. Nakayama, Proc. Imp. Acad. Tokyo **18**, 555–560 (1942); **19**, 144–147 (1943); K. Yosida, ibid. **19**, 356–359 (1943); these Rev. **7**, 253] also uses the Gelfand-Naĭmark theorem, but subsequently employs ideas of partial ordering rather than the Riesz theorem in order to obtain the resolution of the identity of a commutative B^*-algebra of operators in Hilbert space.]

Chapter I is devoted to the finite-dimensional case for purposes of orientation and motivation. Chapter II contains (without proof) those facts about Banach spaces which are used in the sequel. Chapter III deals with elementary facts about spectra and resolvents of operators, with analytic functions with values in a Banach space, their integrals, Taylor's expansions, and immediate consequences. Chapters IV–VI then treat in more detail with Banach algebras, culminating in the above mentioned Gelfand-Naĭmark representation theorem of B^*-algebras based on the Stone-Weierstrass approximation theorem [see M. H. Stone, Trans. Amer. Math. Soc. **41**, 375–481 (1937)] which is proved in Chapter V. Chapter VI ends with an application of the above to the Stone-Čech compactification of a completely regular space [see M. H. Stone, loc. cit.]. This is a very convenient, elegant, and complete exposition of the facts relevant to the representation theorem. [The theorem was first proved by Gelfand and his collaborators; see Gelfand, Mat. Sbornik (N.S.) **9**, 3–24 (1941); these Rev. **3**, 51; Gelfand and Šilov, ibid., 25–39 (1941); these Rev. **3**, 52; Gelfand and Naĭmark, ibid., **12**, 197–213 (1943); these Rev. **5**, 147. See also the expository article in Russian by I. M. Gelfand, D. A. Raĭkov, and G. E. Šilov, Uspehi Matem. Nauk (N.S.) **1**, no. 2(12), 48–146 (1946); these Rev. **10**, 258; and the paper by Kadison reviewed below.] Chapter VII deals with the Hildebrandt-Fichtenholz-Kantorovič theorem on the representation of bounded functionals on the space of all bounded (real or complex) functions on a set by means of integrals with respect to finitely additive set functions [see G. M. Fichtenholz and L. V. Kantorovitch, Studia Math. **5**, 69–98 (1935); A. D. Alexandrov, Mat. Sbornik (N.S.) **8**, 307–348 (1940); **9**, 563–628 (1941); these Rev. **2**, 315; **3**, 207], and the Riesz theorem mentioned in the second paragraph of this review [which in this generality is not to be found in print so far as the reviewer is aware; see also P. R. Halmos, Measure Theory, Van Nostrand, New York, 1950, pp. 247 and 249; these Rev. **11**, 504]. Chapter VIII applies the above to spectral theory by exploiting the isomorphism of the B^*-algebra A of operators in some Hilbert space H with the ring $C(\Lambda)$ of all continuous functions on a compact space Λ as follows: Let $f = f(\lambda)$ be an element of $C(\Lambda)$, $T(f)$ the corresponding operator in A. The inner product $(T(f)x, y)$ for x and y in H is a linear functional on $C(\Lambda)$; hence, by the Riesz theorem, there exists a corresponding regular (possibly complex-valued) measure on Λ, call it $\mu(e, x, y)$ where e is a Borel subset of Λ, such that $(T(f)x, y) = \int_\Lambda f(\lambda)d\mu(\lambda, x, y)$. If $\mu(e, x, y)$ is considered as a function of x and y, it is a bilinear hermitean symmetric form, hence expressible as $\langle E_e x, y\rangle$ where E_e is a projection. The resolution of the identity consists then of the E_e. The facts about the functional calculus follow from the Riesz representation:

$$(T(f)x, y) = \int_\Lambda f(\lambda)d\mu(\lambda, x, y) = \int_\Lambda f(\lambda)d(E_\lambda x, y).$$

G. K. Kalisch (Minneapolis, Minn.).

Kadison, Richard V. A representation theory for commutative topological algebra. Mem. Amer. Math. Soc., no. **7**, 39 pp. (1951).

This paper is devoted to the study of the functional representation method and the unification of theorems developed in recent years after the early work of Stone, Gelfand, Kakutani and others on the commutative spectral theory. They center around the characterization of the space $C(X)$ of all real or complex-valued continuous functions on a compact X and rest on the availability of the Tychonoff, Stone-Weierstrass and Krein-Milman theorems. The author starts out (§2) with the following general result. Theorem 2.1: Let V be an Archimedian ordered vector space with an order unit. Then the natural map of V into the real $C(X)$, where X is the compact weak closure of the set of all extreme maximal ideals of V, is an isomorphism with respect to the units. Here the representation space is not the too big set of all maximal ideals, so the result applies at once to the next cases. In §3 the Stone algebra theorem is derived from the preceding. Theorem 3.1: Let A be a real algebra which is an Archimedian ordered vector space with unit (in both senses), complete in the natural norm, where $x \geqq 0$, $y \geqq 0$ imply $xy \geqq 0$. Then A is isomorphic to a real $C(X)$. The algebra A is not assumed commutative or associative nor are squares required to be positive; these facts follow from the others. §4 derives the Kakutani-Krein lattice theorem from Theorem 2.1 by showing that the natural map is now onto and that the set of extreme maximal ideals is already closed. §5 treats the Banach space characterization of a real $C(X)$ due to Arens and Kelley by methods emphasizing the order situation. The long §5 is devoted to proving some real Banach algebra representation theorems by purely real

methods and to passing from these to the basic real and complex theorems of Gelfand. In doing so the analytic functions techniques are not avoided although a version of the real Mazur-Gelfand theorem is proved by non-complex methods. Theorem 6.3: Let A' be a real Banach algebra ($\|xy\| \leqq \|x\| \cdot \|y\|$) containing a dense subalgebra A with unit 1, $\|1\| = 1$, such that $(x^3+1)^{-1}$ exists in A' for $x \in A$ and $(x^3-1)^{-1}$ does not exist in A' for $x \in A$, $\|x\| = 1$. Then A' is isomorphic to a real $C(X)$. The author points out that the theorem as it stands with conditions on the incomplete A is needed later on. Theorem 6.3 is used to show that a real Banach algebra with unit 1, $\|1\| = 1$, where $\|x^2\| = \|x\|^2$ and $(x^3+1)^{-1}$ always exists is isomorphic to a real $C(X)$ and to get the important representation as a complex $C(X)$ of a commutative B^*-algebra (complex Banach algebra with map $x \to x^*$ such that $(x+y)^* = x^*+y^*$, $(\lambda x)^* = \bar{\lambda} x^*$, $x^{**} = x$, $(xy)^* = y^*x^*$, $\|xx^*\| = \|x\| \cdot \|x^*\|$) with unit. §7 reduces any polynomial identity on the norm to that of Gelfand. Finally, §8 applies these results to commutative C^*-algebras (uniformly closed self-adjoint algebras) of operators in a Hilbert space. Most of the proofs in this paper are neater than those already existing in the fairly complete list of references at the end.

L. Nachbin (Rio de Janeiro).

Tate, John. On the relation between extremal points of convex sets and homomorphisms of algebras. Comm. Pure Appl. Math. **4**, 31–32 (1951).

Let A be a commutative algebra with unit over the real field R, such that for each $a \in A$, $1+\epsilon a$ is a square whenever $|\epsilon|$ is sufficiently small. Let K be the convex set of linear functionals λ on A such that $\lambda(a^2) \geqq 0$ for all $a \in A$ and $\lambda(1) \leqq 1$. By an elementary argument, the author proves that the extreme points of K are precisely the homomorphisms of A into R (i.e., the multiplicative linear functionals on A). [For a related result see Lemma 6.9 of R. V. Kadison's paper reviewed above.] *V. L. Klee, Jr.*

Feldman, Chester. The Wedderburn principal theorem in Banach algebras. Proc. Amer. Math. Soc. **2**, 771–777 (1951).

Soit A une algèbre de Banach, R son radical (au sens de Jacobson-Chevalley). L'auteur se propose de chercher des conditions moyennant lesquelles A est somme directe topologique de R et d'une sous-algèbre S isomorphe (topologiquement) à A/R. Il montre par un exemple qu'il ne suffit pas pour cela que R soit de dimension finie, mais que le théorème est vrai si A/R est de dimension finie. Lorsque A/R n'est plus de dimension finie, mais que R est de dimension finie, on peut encore obtenir le résultat cherché en assujettissant A/R à être isomorphe à certains types d'algèbres introduites par I. Kaplansky [Duke Math. J. **16**, 399–418 (1949); ces Rev. **11**, 115], par exemple les C^*-algèbres complètement continues. *J. Dieudonné.*

Kaplansky, I. Topological algebra. Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 112–113. Amer. Math. Soc., Providence, R. I., 1952.

A brief summary of the author's report to the International Congress of Mathematicians, 1950. A detailed account will appear elsewhere.

Birkhoff, Garrett. Moyennes des fonctions bornées. Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 143–153. Centre National de la Recherche Scientifique, Paris, 1950.

Let A be a linear, associative ring with unit 1, with commutative multiplication and admitting multiplication by scalars. (In §3 the author seems to state that such a ring A must be isomorphic to a function space if it has a finite basis and $f^2 = 0$ implies $f = 0$ but this need not hold even when A has dimension 2.) The author calls a linear operator a Reynolds operator if $T(1) = 1$ and $T(Tf \cdot g) = Tf \cdot Tg$ for all f, g in A. Properties of such T are discussed. All such T are determined explicitly for the case that A consists of all functions defined at a finite set of points. Some results are also obtained for other function spaces such as the set of all continuous functions over a compact space. For the abstract case, with A a ring lattice and T satisfying the condition $Tf \geqq 0$ whenever $f \geqq 0$ a subdirect decomposition theorem is proved. *I. Halperin* (Kingston, Ont.).

Kantorovič, L. V., Vulih, B. Z., and Pinsker, A. G. Partially ordered groups and linear partially ordered spaces. Uspehi Matem. Nauk (N.S.) **6**, no. 3(43), 31–98 (1951). (Russian)

This paper is a survey of the present status of the theory of lattice-ordered groups and linear spaces, with special emphasis on those in which the lattice ordering is complete. Most of the material considered is also found in the authors' Funkcional'nyĭ analiz v poluuporyadočennyh prostranstvah [Gosudarstv. Izdat. Tehn.-Teor. Lit. Moscow-Leningrad, 1950; these Rev. **12**, 340]. An extensive and truly ecumenical bibliography is appended; for many results, complete and accurate historical references are given. *E. Hewitt.*

Nakamura, Masahiro. Notes on Banach space. XI. Banach lattices with positive bases. Tôhoku Math. J. (2) **2**, 135–141 (1950).

Throughout E is a separable Banach lattice and E^* is the conjugate space. The writer takes up some questions in which the condition A: every lattice segment $\{y \mid 0 < y < x\}$ is norm compact, plays a central role and the proofs depend on the Gelfand-Phillips theorem [Gelfand, Mat. Sbornik N.S. **4**(46), 235–284 (1938); Phillips, Trans. Amer. Math. Soc. **48**, 516–541 (1940); these Rev. **2**, 318]. Thus, for instance, A is necessary and sufficient for the w^* (sequential) continuity of the lattice operations on E^* as well as for the simultaneous lattice completeness and existence of positive orthogonal base in E and the implication $x_\alpha \downarrow \theta$ implies $\|x_\alpha\| \to \theta$. *D. G. Bourgin* (Urbana, Ill.).

Ogasawara, Tôzirô. Some general theorems and convergence theorems in vector lattices. J. Sci. Hiroshima Univ. Ser. A. **14**, 14–25 (1949).

The properties of conjugate spaces and reflexivity of Banach lattices are studied. (The conjugate space $\bar{X}$ of a vector lattice X is defined as the set of its bounded linear functionals.) It is shown that a Banach lattice is reflexive if and only if one of the following conditions holds: (i) it is locally compact, (ii) it is weakly complete, together with its conjugate space. If its second conjugate space is separable, then it is reflexive. Various other results are proved. *G. Birkhoff* (Cambridge, Mass.).

Nakano, Hidegorô. Modulared linear spaces. J. Fac. Sci. Univ. Tokyo. Sect. I. 6, 85–131 (1951).

Let $\mathfrak{M}$ be a linear space of vectors x. A modular on $\mathfrak{M}$ (more general than a norm) is a function $m(x)$ such that $0 \leqq m(x) \leqq \infty$, $m(x)=m(-x)$, $m(\lambda x)=\infty$ for all positive λ does not occur for any x, $m(\lambda x)=0$ for all positive λ occurs only for $x=0$, $m(x)=\sup\{m(\lambda x);\ 0 \leqq \lambda < 1\}$, and $m[\alpha x+(1-\alpha)y] \leqq \alpha m(x)+(1-\alpha)m(y)$ whenever $0 \leqq \alpha \leqq 1$. Vectors a_n are said to be modular convergent if for each λ, $m[\lambda(a_n-a_p)] \to 0$ as $n, p \to \infty$ and modular convergent to a if for each λ, $m[\lambda(a-a_n)] \to 0$ as $n \to \infty$. A linear functional $\bar{x}(x)$ is called modular bounded if the $\bar{x}(x)$ with $m(x) \leqq 1$ are bounded, equivalently if there are constants A, B such that $\bar{x}(x) \leqq A+Bm(x)$ for all x. The existence of modular bounded functionals is derived from an extension theorem of the Hahn-Banach type and the linear space $\overline{\mathfrak{M}}$ of all such x is assigned the modular $\bar{m}(\bar{x})=\sup[\bar{x}(x)-m(x)]$ (note that if $m(\lambda x)=|\lambda| m(x)$ for all λ, x, the only possible values of $\bar{m}(\bar{x})$ are 0 and ∞). The inequality $\bar{x}(x) \leqq \bar{m}(\bar{x})+m(x)$ holds. It is shown that $\overline{\mathfrak{M}}$ is modular complete and that every x in $\mathfrak{M}$ can be identified with an element in $\overline{\overline{\mathfrak{M}}}$ with $\overline{\bar{m}}(x)=m(x)$. Thus $\mathfrak{M}$ can be considered as imbedded in the modular complete $\overline{\overline{\mathfrak{M}}}$.

Let $\bar{S}$ be any non-empty set of $\bar{x}$ such that

$$\varphi(x)=\sup\{\bar{x}(x);\ \bar{x} \text{ in } \bar{S}\}$$

is finite for each x in $\mathfrak{M}$. Then $\varphi(x)$ is of course a norm. The first norm, $\|x\|$, is defined as $\sup\{\bar{x}(x);\ \bar{m}(\bar{x}) \leqq 1\}$ and the second norm, $|||x|||$, as $\sup\{\bar{x}(x);\ \|\bar{x}\| \leqq 1\}$. Then $|||x||| \leqq \|x\| \leqq 2|||x|||$. With either norm, it is shown that modular convergence for vectors a_n is equivalent to norm convergence and for linear functionals modular boundedness is equivalent to norm boundedness. Thus the spaces $\mathfrak{M}$ and $\overline{\overline{\mathfrak{M}}}$ can be identified with the Banach-conjugate spaces.

$\mathfrak{M}$ is called simple if $m(x)=0$ only for $x=0$, uniformly simple if $\inf\{m(\lambda x);\ m(x) \geqq 1\}>0$ for each positive λ, uniformly monotone if $\lambda \sup\{m(x/\lambda);\ m(x) \leqq 1\} \to 0$ as $\lambda \to \infty$. Uniform simplicity and uniform monotony are shown to be conjugate properties; if $\mathfrak{M}$ has one of them $\overline{\mathfrak{M}}$ has the other. $\mathfrak{M}$ is called finite if $m(x)$ is finite for every x, uniformly finite if $\sup\{m(\lambda x);\ m(x) \leqq 1\}$ is finite for each λ, uniformly increasing if $|\lambda| \inf\{m(x/\lambda);\ m(x) \geqq 1\} \to \infty$ as $\lambda \to 0$. To be uniformly finite and to be uniformly increasing are shown to be conjugate properties.

$\mathfrak{M}$ is said to be of uniformly convex modular if for positive γ, ϵ there is a positive δ such that $m(a), m(b) \leqq \gamma$, $m(a-b) \geqq \epsilon$ imply $\frac{1}{2}\{m(a)+m(b)\} \geqq m(\frac{1}{2}(a+b))+\delta$, and of uniformly even modular if $m(a), m(b) \leqq \gamma$, $\|a-b\| \leqq \delta$ imply

$$\tfrac{1}{2}\{m(a)+m(b)\} \leqq m(\tfrac{1}{2}(a+b))+\epsilon\|a-b\|.$$

If $\mathfrak{M}$ is uniformly simple, uniformly increasing and of convex modular then $\overline{\mathfrak{M}}$ is of uniformly even modular; if $\mathfrak{M}$ is uniformly finite, uniformly increasing and of uniformly even modular, then $\overline{\mathfrak{M}}$ is of uniformly convex modular. If $\mathfrak{M}$ is uniformly simple, modular complete and of uniformly convex modular then $\mathfrak{M}=\overline{\mathfrak{M}}$. These ideas are studied in the special case that $m(x)$ is a norm and for a general modular in terms of its first norm. If X is an element of the quotient space of $\mathfrak{M}$ relative to one of its linear subspaces, the modular $m(X)$ is defined as $\sup[\{\inf m(\lambda x);\ x \text{ in } X\};\ 0 \leqq \lambda < 1]$. Properties of $m(X)$ are derived from corresponding properties of $m(x)$. *I. Halperin* (Kingston, Ont.).

Nakano, Hidegorô. Spectral theory and its application in continuous linear lattices in which a product is defined. Sûgaku (Mathematics) 1, 77–88 (1948). (Japanese)

The results of this paper were later incorporated in the author's book, Modern Spectral Theory [Maruzen, Tokyo, 1950; these Rev. 12, 419]. *K. Yosida* (Nagoya).

Nakano, Hidegorô. The individual ergodic theorem in vector lattices. Sûgaku (Mathematics) 1, 257–263 (1949). (Japanese)

The results were published later in Ann. of Math. (2) 49, 538–556 (1949); these Rev. 10, 550. *K. Yosida.*

Varsavsky, Oscar Alberto. The ergodic theorem in quantum mechanics. Revista Unión Mat. Argentina 14, 350–365 (1950). (Spanish. English summary)

The author considers the discrete group of transformations formed of the powers T^n of a linear transformation T on a finite-dimensional Hilbert space. An ergodic theorem is established and interpreted in terms of limiting probability of a Markoff chain of measurements. *W. Kaplan.*

Theory of Probability

Fréchet, Maurice. Lectures on random elements of arbitrary nature. Trabajos Estadistica 1, 157–181 (1950). (Spanish)

Expository. The emphasis is on typical values determined by a distribution on an abstract space. *J. L. Doob.*

Guillaumin, Gustave. Les espaces probabilistes. Rev. Gén. Sci. Pures Appl. N.S. 58, 198–205 (1951).

Expository article [cf. Menger, Proc. Nat. Acad. Sci. U. S. A. 37, 178–180, 226–229 (1951); C. R. Acad. Sci. Paris 232, 2001–2003 (1951); these Rev. 13, 51].

K. L. Chung (Ithaca, N. Y.).

Ríos, S. On convergence of distributions and convergence in probability. Revista Mat. Hisp.-Amer. (4) 11, 61–64 (1951). (Spanish)

Ríos, Sixto. On convergence of distributions and convergence in probability. Trabajos Estadistica 2, 75–78 (1951). (Spanish. English summary)

***Zygmund, A. A remark on characteristic functions.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 369–372. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

Let $F_1, F_2, \cdots$ be distribution functions in one variable, with corresponding characteristic functions $\phi_1, \phi_2, \cdots$. Suppose that the sequence of characteristic functions converges in some interval about the origin, and that the limit is continuous at the origin. Then if (*) $F_n(t)=0$ for $t \leqq 0$, there is a distribution function F such that $F_n \to F$ at every continuity point of F. The theorem is known to be false without (*), without which it is usually supposed that the ϕ_n-sequence converges for all values of the argument. Proofs are sketched that the corresponding theorem for k-dimensional distribution functions is true, and that the condition (*) can be replaced by the weaker condition that, for some $A>0$, $\epsilon>0$ t_0, $F_n(t) \leqq Ae^{-\epsilon|t|}$ if $t \leqq t_0$. *J. L. Doob* (Urbana, Ill.)

Mihoc, Gh. La loi limite de la probabilité des nombres des itérations de longueur donné. An. Acad. Repub. Pop. Române. Secț. Ști. Mat. Fiz. Chim. Ser. A. 2, 213–227 (1949). (Romanian. Russian and French summaries)

The author studies the asymptotic distribution of the number of runs of length m in a sequence of n independent trials, finding, as is known, that it is Gaussian or Poisson according to the way m varies as $n\to\infty$ [see v. Mises, Z. Angew. Math. Mech. 1, 298–307 (1921)]. *J. L. Doob.*

Hoeffding, Wassily. A combinatorial central limit theorem. Ann. Math. Statistics 22, 558–566 (1951).

Let $c_n(i,j)$, $i,j=1,\cdots,n$, be n^2 real numbers, defined for every positive integer n, and set

$$d_n(i,j)=c_n(i,j)-\frac{1}{n}\sum_{j=1}^{n}c_n(g,j)-\frac{1}{n}\sum_{h=1}^{n}c_n(i,h)+\frac{1}{n^2}\sum_{g=1}^{n}\sum_{h=1}^{n}c_n(g,h).$$

Let $(Y_{n1},\cdots,Y_{nn})$ be a random vector which takes on the $n!$ permutations of $(1,\cdots,n)$ with equal probabilities. Then the distribution of $S_n=\sum_{i=1}^{n}c_n(i,Y_{ni})$ is asymptotically normal with

$$ES_n=\sum_{i=1}^{n}\sum_{j=1}^{n}c_n(i,j)/n \quad\text{and}\quad \operatorname{var}S_n=\sum_{i=1}^{n}\sum_{j=1}^{n}d_n^2(i,j)/(n-1)$$

if

$$\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}d_n^r(i,j)}{\left[\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}d_n^2(i,j)\right]^{r/2}}=0,\quad r=3,4,\cdots.$$

This condition is satisfied if

$$\lim_{n\to\infty}\frac{\max_{1\le i,j\le n}d_n^2(i,j)}{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}d_n^2(i,j)}=0.$$

The conditions are simplified in the particular case

$$c_n(i,j)=a_n(i)b_n(j).$$

G. E. Noether (Boston, Mass.).

Feller, William. The problem of n liars and Markov chains. Amer. Math. Monthly 58, 606–608 (1951).

Eddington's problem of four liars is shown to be, and solved as, a problem in Markoff chains. Variations of this problem: several lies, preferential lying, variable proneness to lie, lead to the general Markoff chain with a finite number of states. *M. Loève* (Berkeley, Calif.).

***Lehman, R. Sherman. A problem on random walk.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 263–268. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

Consider tossings of a uniform l-faced die and call an equalization a place in the sequence of tossings where each face has appeared equally often. It is shown that the probability of an equalization is 1 for $l=2$, 3, but less than 1 otherwise. A scheme for translating this into the language of random walks is given for $l=2$, 3 and 4, and it is noted that an equalization corresponds to a return to equilibrium. Finally it is shown that the probability of reaching any point of the lattice of random walks has the same general character. As noted by the author, the subject has already been treated in Feller's treatise [An introduction to probability theory . . . , vol. 1, Wiley, New York, 1950; these Rev. 12, 424]. *J. Riordan* (New York, N. Y.).

Good, I. J. Random motion on a finite Abelian group. Proc. Cambridge Philos. Soc. 47, 756–762 (1951).

Generalization of random walk with absorbing points to a finite Abelian group. By representing the group as the direct product of cyclic groups Fourier transforms are introduced. By means of these the generating functions for absorption probabilities are studied in much the same way as in the ordinary random walk on the integers.

K. L. Chung (Ithaca, N. Y.).

Lévy, Paul. La mesure de Hausdorff de la courbe du mouvement brownien à n dimensions. C. R. Acad. Sci. Paris 233, 600–602 (1951).

Let $\varphi(\rho)$ be a positive valued function defined for $\rho>0$ such that (i) $\varphi(\rho)\downarrow 0$ as $\rho\downarrow 0$ and (ii) $\lim \varphi(\rho)/\rho^2\log\log(1/\rho)=c$ exists, where c can be 0 or $+\infty$. Let $x(t,\omega)$ be an n-dimensional Brownian motion, let $E(I,\omega)=\{x(t,\omega)\,|\,t\in I\}$ be the path of $x(t,\omega)$ corresponding to the interval I of real numbers t, and let $\rho(I,\omega)$ be the diameter of $E(I,\omega)$. Let, further, $I_{h,k}=\{t\,|\,k/2^h\le t\le(k+1)/2^h\}$, $h,k=0,1,2,\cdots$. The author states, without proof, that for any positive number T and for almost all ω, the limit

$$\lim_{h\to\infty}\sum_{0\le k\le[2^hT]}\varphi(\rho(I_{h,k},\omega))$$

exists and is equal to $c\lambda_nT$, where $[2^hT]$ is the integral part of 2^hT and λ_n is the smallest positive zero point of the solution $u(s)$ of the differential equation

$$2s\frac{d^2u}{ds^2}+n\frac{du}{ds}+u=0$$

which is holomorphic at $s=0$. The author applies this result to show that the Hausdorff φ-measure of the path of a Brownian motion $x(t,\omega)$ is zero for almost all ω if $c=0$, i.e. if $\varphi(\rho)=o(\rho^2\log\log(1/\rho))$ and, in particular, if $\varphi(\rho)=\rho^2$. The author also conjectures that, under the conditions (i), (ii) above, the Hausdorff φ-measure of $E(I,\omega)$ for $I=\{t\,|\,0\le t\le T\}$ is equal to $c\lambda_n'T$ if $n>2$, where λ_n' is a constant depending only on the dimension n, while in case $n=2$ the function $\rho^2\log\log(1/\rho)$ in the condition (ii) must be replaced by $\rho^2\log(1/\rho)\log\log(1/\rho)$. *S. Kakutani.*

Itô, Kiyosi. On a formula concerning stochastic differentials. Nagoya Math. J. 3, 55–65 (1951).

In the following, t ranges over a given finite interval, and ω is a probability parameter, ranging over a (probability) measure space. Subscripts and superscripts range over the integers $1,\cdots,n$. Let $\{\xi^i(t,\omega)\}$, $\{a^i(t,\omega)\}$, $\{b_j^i(t,\omega)\}$, $\{\beta^j(t,\omega)\}$ define families of random variables (stochastic processes). It is supposed that the n-dimensional β-process is a Brownian motion, and that, roughly, β-increments are independent of earlier ξ, a, b, and β-process random variables. The equation system

$$d\xi^i(t,\omega)=a^i(t,\omega)dt+\sum_j b_j^i(t,\omega)d\beta^j(t,\omega)$$

represents the system obtained from it by formal integration. If this equation system is true, and if

$$\eta(t,\omega)=f[t,\xi^1(t,\omega),\cdots,\xi^n(t,\omega)],$$

where f is a function satisfying specified regularity hypotheses, then it is shown that

$$d\eta(t,\omega)=g(t,\omega)dt+\sum_j h_j(t,\omega)d\beta^j(t),$$

where g and h_j are given explicitly in terms of a^i, b_j^i, ξ^i and the first and second partial derivatives of f. This result was used without detailed proof in an earlier paper by the author [Nagoya Math. J. 1, 35–47 (1950); these Rev. 12, 425].

J. L. Doob (Urbana, Ill.).

Itô, Kiyosi. Multiple Wiener integral. J. Math. Soc. Japan 3, 157–169 (1951).

Let T be a measure space, and let A be a subset of T of finite measure $m(A)$. For each A, let $\beta(A)$ be a Gaussian random variable with mean 0, and suppose that $E\{\beta(A)\beta(A')\}=m(AA')$. The author defines the multiple integral

$$I(f)=\int\cdots\int f(t_1,\cdots,t_p)\prod_1^p d\beta(t_j).$$

The definition makes

$$E\{|I(f)|^2\}=p!\int\cdots\int|f_1|^2\prod_1^p m(dt_j),$$

where f_1 is the average of the transforms of t under the $p!$ permutations of its arguments. Moreover, if H_p is the Hermite polynomial of degree p, and if ϕ_1, ϕ_2 are orthogonal on T,

$$\int\cdots\int\prod_1^{p_1}\phi_1(s_i)\prod_1^{p_2}\phi_2(t_j)d\beta(s_i)d\beta(t_j)=\prod_1^{\infty}2^{-p_i/2}H_{p_i}\left(\frac{1}{\sqrt{2}}\int\phi_i(t)d\beta(t)\right).$$

In particular, if T is the line and m is Lebesgue measure, $\beta(A)$ defines the usual Brownian motion, and $I(f)$ can be evaluated as an iterated integral, using the stochastic integral defined by the author in a previous paper [Proc. Imp. Acad. Tokyo 20, 519–524 (1944); these Rev. 7, 313]. The orthogonal development of functions on a Brownian motion process, discussed by Cameron and Martin [Ann. of Math. (2) 48, 385–392 (1947); these Rev. 8, 523] goes over without change to the general case considered here. The author's multiple integral differs slightly from that given by Wiener [Amer. J. Math. 60, 897–936 (1938)].

J. L. Doob.

Bartlett, M. S. The dual recurrence relation for multiplicative processes. Proc. Cambridge Philos. Soc. 47, 821–825 (1951).

The author discusses various aspects of the treatment of continuous, as contrasted with discrete, parameter multiplicative processes, stressing the relations between forward and backward equations for the usual generating functions, and their domains of applicability.

J. L. Doob.

Messel, H. On the fluctuation of a nucleon cascade in homogeneous nuclear matter and calculation of average numbers. I. Proc. Roy. Irish Acad. Sect. A. 54, 125–135 (1951).

Starting from the nuclear cascade equation of Jánossy [Proc. Phys. Soc. Sect. A. 63, 241–249 (1950)] the author derives the following pair of equations for the mean number of nucleons, $\bar{N}(\epsilon,x)=N(\epsilon,x)$ (say) of energy ϵ at depth x and the mean square deviation $T(\epsilon,x)=\bar{N}^2(\epsilon,x)-N(\epsilon,x)$:

$$\frac{\partial}{\partial x}N(\epsilon,x)+\alpha N(\epsilon,x)=\int_0^\infty N\left(\frac{\epsilon}{\epsilon'},x\right)W(\epsilon')d\epsilon'$$

and

$$\frac{\partial}{\partial x}T(\epsilon,x)+\alpha T(\epsilon,x)=\int_0^\infty T\left(\frac{\epsilon}{\epsilon'},x\right)W(\epsilon')d\epsilon'+F(\epsilon,x),$$

where α is a constant,

$$W(\epsilon)=\int_0^\infty[w(\epsilon,\epsilon')+w(\epsilon',\epsilon)]d\epsilon'=\int_0^\infty\mathfrak{w}(\epsilon,\epsilon')d\epsilon',$$

$$F(\epsilon,x)=\int_0^\infty\int_0^\infty N\left(\frac{\epsilon}{\epsilon'},x\right)N\left(\frac{\epsilon}{\epsilon''},x\right)\mathfrak{w}(\epsilon',\epsilon'')d\epsilon'd\epsilon''$$

and

$w(\epsilon',\epsilon'')=\text{constant}\times\epsilon''^{\beta}(1-\epsilon')^{\gamma}$ (β, γ positive constants). The equations for $N(\epsilon,x)$ and $T(\epsilon,x)$ are solved by the use of Mellin transformations. The solutions are illustrated by numerical examples.

S. Chandrasekhar.

* **Morse, Philip M., and Kimball, George E. Methods of Operations Research.** The Technology Press of Massachusetts Institute of Technology; John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. vii+158 pp. $4.00.

"Operations research is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control." With this carefully framed definition the authors, pioneers in the field, set forth the bounds of the scientific activity which scored so many brilliant and exciting triumphs in World War II. It is their purpose to indicate the techniques used by OR (Operations Research) and to illustrate by examples the use of these techniques. The book was first published in classified form after the war. Most of the examples are drawn from military applications, and many parts of the book were written during and at the end of the war by various persons in the Operations Research Group, U. S. Navy.

The word "operation" carries the connotation of repetitive action, which in turn implies the prime importance of statistics and probability to OR. Chapter 2 is apparently intended as a review to highlight those parts of the theory which are of greatest use in this work. It begins with the fundamental concepts and leads rapidly through the simpler distributions to a short section on sampling. The authors adopt an unorthodox definition of a random variable ξ as one "which will have any value (within its allowed range) with equal probability." A stochastic variable x is then defined as a function of ξ. The treatment throughout is highly intuitive, becoming, at times, inaccurate, misleading, or downright wrong [e.g., "The standard deviation is, of course, the expected value of the root-mean-square deviation."] The text is lively, though, and there are many illuminating examples accompanied by good practical advice. [In the reviewer's opinion, the overall effectiveness of an OR group would be increased by a longer, more rigorous course of study in probability and statistics for its new members; perhaps this question should be made the subject of an OR study!]

In the first analysis of a problem, one decides on the "constants of the operation," the measures of effectiveness. These are to be computed approximately, neglecting details, and are to be compared with the theoretical optimum, if available. At first large factors of improvement are sought, so large that a logarithmic scale is frequently used, and "hemibel thinking" is encouraged. The variation of a measure of effectiveness often gives a clue to changing conditions or tactics, and can provide a graphic history of the operation

over a period of time. Chapter 3 contains numerous examples of the sort of constants which can be looked for, and of the sort of conclusions which may be derived from their study. These include sweep rates in search (number of contacts per unit searching effort per unit target density) and exchange rates (ratio of enemy loss to own loss). One section is devoted to the problem of the relative effectiveness of two different weapons or tactics in gaining some strategic end, and another to the evaluation of equipment performance. It is pointed out that equipment should be used properly, and that slight modifications may increase its effectiveness considerably. Maintenance, training, and supervised practice are shown to be of great importance. One is impressed again and again by the simplicity of the methods used to obtain large improvements in the operations, and one cannot help recalling the earlier advice of the authors not to place too much reliance on the "expert opinion" of some of the users of equipment. This chapter is indeed most enlightening and instructive.

The next two chapters, on strategical kinematics and tactical analysis, are of considerable theoretical interest. Here, heavier artillery is brought to bear upon problems that will not yield to simple statistical analysis. There is a discussion of various forms of the Lanchester differential equations relating the losses of opposing forces, similar to those which occur in the analysis of chemical reactions or of struggle between animal species. There is also an analysis of the problem of the circulation of submarines in the Atlantic, showing the effects on this circulation of three distinct types of offensive action: hunting them in the Atlantic, bombing their repair bases, and bombing the factories where they are built. Typical solutions are presented. Under tactical analysis, there is a statistical treatment of the problems of suicide planes and of submarine casualties, and a number of examples of analytical solutions involving search theory. Particular stress is laid on the "mean free path theorem," concerning the probability of detection when the search effort is distributed at random, or its extension, that the number of detections (or hits) has the Poisson distribution. The last section deals with measures and countermeasures, leading into problems involving the theory of games.

The gunnery and bombardment problems treated in chapter 6 are admittedly not in the province of the OR worker, but the authors feel that he should be familiar with the techniques of evaluating weapon performance, of calculating probabilities of destruction, and of determining patterns which create maximum destruction. [The reviewer was surprised at the following statement: "This chapter is included, as was chapter 2, because it is believed that ability in the broader problems is improved by the mastery of the narrower technical details." To regard probability theory, which plays so fundamental a role in OR work, in the same light as gunnery seems an obvious error.]

Chapter 7 (Operational experiments with equipment and tactics) contains a great deal of very sound advice on planning such experiments, including the necessity for rapid on-the-spot reduction of data and for a continuing, though preliminary, analysis of results. The specific problems of accuracy measurements on weapons, evaluation of detection equipment, and survival problems in gunnery are discussed.

The last chapter deals with the organizational and procedural problems of an OR group in a military service, but many of the conclusions reached are of wider validity. There is a need for broad knowledge in the group, but little power; the group (or its individual member) acts only as an advisor to the officer in command. It is important to have contacts with several echelons, and to be able to cut across command boundaries for rapid formulation of problems and dissemination of solutions. It is frequently necessary to have subgroups in the field, the personnel circulating between the sub-group and headquarters. Their purposes are to assist service units, to secure information for transmission to the parent group, and to give the individual a practical background.

There is a table of random digits, one of random angles, and one of random normal deviates, to be used in computing probabilities by sampling; there are also tables of the binomial, normal, and Poisson distributions. The typography, with few exceptions, is excellent, and the format attractive. Bibliography and index seem quite adequate, except for the inclusion of only one reference book on probability (Fry's Probability and its engineering uses [van Nostrand, New York, 1928]) and a mysterious reference in the text (p. 135) to something called Division 6, Volume 2B, presumably a treatise on search theory.

Reviewer's Opinion. The book, in its original form, undoubtedly served the valuable purpose of indoctrinating and training new members of the U. S. Navy Operations Research Group, and in its present form it achieves its goal of acquainting scientists and engineers with the accomplishments of OR, and of giving them some insight into its methods. Nevertheless, the claim that a new applied science has come into being does not seem to be completely validated by the authors' treatment. No collection of brilliant examples of scientific thinking could accomplish this without a process of distillation of general principles. That these exist and are of wide applicability will hardly be questioned, and indeed many of them are evident in scattered and partially revealed form throughout the book. An explicit and orderly statement of such principles, and a formulation of general problems, could lead to fruitful investigations in the theory of operations research, and ultimately to a systematic treatment of large classes of specific problems.

N. J. Fine (Philadelphia, Pa.).

Vaulot, Émile. Les formules d'Erlang et leur calcul pratique. Ann. Télécommun. 6, 279–286 (1951).

Putting $y_x = e^{-y}y^x/x!$ and $Y_x = y_0 + y_1 + \cdots + y_x$ the Erlang formula in question (for the probability of all trunks busy in a simple trunk group of x trunks with exponential holding time and random call input with mean y, and "lost calls cleared") may be written y_x/Y_x. Tables have been published by Conny Palm [Fritzes Hovbokhandel, Stockholm, 1947; these Rev. 9, 207].

The author gives approximate formulas of several kinds for the denominator, which is, of course, the Poisson summation. Two of these are

$$Y_x \approx \Phi(\lambda) + \varphi(\lambda)\left[\frac{f_1(\lambda)}{x^{1/2}} + \frac{f_2(\lambda)}{x} + \frac{f_3(\lambda)}{x^{3/2}} + \cdots\right],$$

$$Y_x \approx \Phi(u),$$

with

$$u = \lambda + \frac{\lambda^2+2}{3x^{1/2}} + \frac{7\lambda^3+5\lambda}{36x} + \frac{219\lambda^4+121\lambda^2-58}{1620x^{3/2}} + \cdots.$$

In these

$$\lambda = (x-y)x^{-1/2}, \quad \varphi(\lambda) = (2\pi)^{-1/2}e^{-\lambda^2/2}, \quad \Phi(\lambda) = \int_{-\infty}^{\lambda} \varphi(z)dz.$$

The second is closely related to G. A. Campbell's formula [Bell System Tech. J. 2, 95–113 (1923)] for the complementary summation; indeed a similar result has been given in a paper by the reviewer [Ann. Math. Statistics 20, 417–425 (1949); these Rev. 11, 41]. A result like the first in terms of a more natural variable (in present notation $(x-y+\frac{1}{2})y^{-1}$) and with a bound for the error has been given by T. T. Cheng [Bull. Amer. Math. Soc. 55, 396–401 (1949); these Rev. 10, 613]. *J. Riordan* (New York, N. Y.).

Mathematical Statistics

Olekiewicz, M. Tables of significance limits for the largest critical ratio out of k ratios. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 4, 115–121 (1950). (French. Polish summary)

Tables are given for the 5% and 1% points for the largest deviate out of k normal deviates and for the 5% point of the largest ratio out of k Student's ratios with ν degrees of freedom; $1 \leqq k \leqq 30$, $1 \leqq \nu \leqq 10$. *H. Chernoff.*

Sukhatme, P. V., Thawani, V. D., Pendharkar, V. G., and Natu, N. P. Revised tables for the d-test of significance. J. Indian Soc. Agric. Statistics 3, 9–23 (1951).

Revised tables are presented for the d-test of significance for the Fisher-Behrens problem. The applicability of these tables is controversial [see the review by A. Wald of R. A. Fisher's paper in Ann. Eugenics 11, 141–172 (1941); these Rev. 3, 175]. *H. Chernoff* (Urbana, Ill.).

***Sato, Ryoichirô. Sûri tôkeigaku. [Mathematical Statistics].** Baifûkan, Tokyo, 1948. 3+600+4 pp.

1. Introduction. 2. Probability theory. 3. Measurable functions and probability functions. 4. Correlation. 5. Statistical theory of tests. 6. Estimation of unknown parameters. Appendix: Statistical estimation and sampling methods. Tables. *Table of contents.*

***Roy, S. N. On some aspects of statistical inference.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 555–564. Amer. Math. Soc., Providence, R. I., 1952.

Expository paper. *J. Wolfowitz* (Ithaca, N. Y.).

Azorín, Francisco. On peakedness and its measure. Trabajos Estadistica 1, 263–272 (1950). (Spanish. English summary)

An expository account, with references, of the use of β_2 as a measure of kurtosis. *H. L. Seal.*

Sakamoto, Heihachi. On independence of statistical quantities. Sûgaku (Mathematics) 1, 263–274 (1949). (Japanese)

Let a variate vector $\xi=(x_1, \cdots, x_n)$ be subject to a normal distribution given by

$$f(\xi)=(2\pi)^{-n/2}|V|^{-1/2}\exp(-(V^{-1}\xi,\xi)/2)d\xi,$$

V denoting a variance matrix of n variates x_i. The author studies the criteria on independency between statistics represented by linear or quadratic forms of samples, with the aid of coefficient vectors or coefficient matrices of those forms. The main result states that for independency between $(A\xi,\xi)$ and $(B\xi,\xi)$, $(A\xi,\xi)$ and $(\mathfrak{A},\xi)$ or $(\mathfrak{A},\xi)$ and $(\mathfrak{B},\xi)$ it is necessary and sufficient that $AVB=0$, $AV\mathfrak{A}=0$ or $(V\mathfrak{A},\mathfrak{B})=0$, respectively; $\mathfrak{A}$ and $\mathfrak{B}$ being coefficient vectors, and A and B symmetric coefficient matrices. An application is made to a deduction of distribution law of sample correlation coefficient from a normal population. *Y. Komatu.*

***Anderson, T. W. The asymptotic distribution of certain characteristic roots and vectors.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 103–130. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

Let A and D be two sample covariance matrices, independently distributed like sums $\sum_{\alpha=1}^{q} y_\alpha y_\alpha'$ where $y_1, \cdots, y_q$ are independent normal vector variables with common covariance matrix. In the case of A the means of the y_α are assumed to be zero; in the case of D the means may not be zero. The paper investigates the asymptotic distribution of the characteristic roots ϕ of $|D-\phi A|$ and the associated characteristic vectors satisfying $Dc=\phi Ac$ as the number of vectors defining A increases indefinitely and the means of the vectors defining D change in a prescribed way. The results are stated in seven theorems which are too long to quote here. *G. E. Noether* (Boston, Mass.).

Hogg, Robert V. On ratios of certain algebraic forms. Ann. Math. Statistics 22, 567–572 (1951).

The author shows that if x and y are random variables, a necessary and sufficient condition for y and x/y to be statistically independent is that, under certain regularity conditions,

$$\frac{\partial^k M(0,t)}{\partial u^k}=\frac{\partial^k M(0,0)/\partial u^k}{\partial^k M(0,0)/\partial t^k}\cdot\frac{\partial^k M(0,t)}{\partial t^k}$$

for $k=0, 1, 2, \cdots$ where $M(u,t)=E(e^{ux+ty})$. Suppose x_i $(i=1, \cdots, n)$ are statistically independent random variables having gamma probability density functions,

$$f_i(x_i)=\frac{1}{\Gamma(c_i+1)d_i^{c_i+1}}(x_i)^{c_i}\exp\left(-\frac{x_i}{d_i}\right) \quad \text{for } x_i \geqq 0,$$

$$f_i(x_i)=0 \quad \text{for } x_i<0,$$

where $c_i>-1$, and $d_i>0$, $i=1, \cdots, n$. It is shown that a necessary and sufficient condition for

$$\sum_1^n b_i x_i \quad \text{and} \quad \sum_1^n a_i x_i / \sum_1^n b_i x_i \qquad (b_i>0)$$

to be statistically independent is that $b_1 d_1=\cdots=b_n d_n$. This paper is motivated by results obtained by Williams, von Neumann, and Koopmans [Ann. Math. Statistics 12, 239–241, 367–395 (1941); 13, 86–88, 14–33 (1942); these Rev. 3, 7; 4, 21, 22] on the ratio of the mean square successive difference to the mean square difference in random samples from a normal distribution. *S. S. Wilks.*

***Welker, E. L. Correlation and regression analysis.** Proceedings, Industrial Computation Seminar, September 1950, pp. 36–43. International Business Machines Corp., New York, N. Y., 1951.

This article presents the usual large sample estimation procedures for both simple and multiple regression; the degrees of freedom are not adjusted for the number of parameters estimated. The author also indicates the use of independent fixed variates, which he might have mentioned as the basis for the Abbreviated Doolittle Method of calculation. *R. L. Anderson* (Raleigh, N. C.).

Bejar, Juan. Remarks on best allocation. Trabajos Estadistica 1, 111–116 (1950). (Spanish)

In sampling n members of a population comprising N_j members in the jth stratum, in order to estimate the mean of a variate (standard deviation σ_j in the jth stratum), an "optimum" procedure is to sample members from the jth stratum ($j=1, 2, 3, \cdots$) proportionate in number to $N_j\sigma_j[\sigma_j/(\sigma_j-1)]^{\frac{1}{2}}$; an approximation to this is to sample proportionate numbers $N_j\sigma_j$ [Kendall, The Advanced Theory of Statistics, vol. 2, Charles Griffin, London, 1946, Ch. 25; these Rev. 8, 473]. The author investigates under what circumstances the approximate numbers differ from the truth by a unit or more in any stratum. *H. L. Seal.*

Votaw, D. F., Jr., Rafferty, J. A., and Deemer, W. L. Estimation of parameters in a truncated trivariate normal distribution. Psychometrika 15, 339–347 (1950).

This paper gives maximum-likelihood estimators for certain parameters in a truncated trivariate normal distribution when the values of the other parameters are known. The estimators are functions of a random sample. Approximate variances and covariances of the estimators, when the sample size is large, are also given. The type of truncation considered is merely restriction of the range of one of the variates, whose true mean and variance are assumed to be known. (From the author's summary.) *H. Chernoff.*

Rushton, S. On a sequential t-test. Biometrika 37, 326–333 (1950).

Consider a normal population with unknown mean μ and unknown standard deviation σ. In testing the hypothesis $\mu/\sigma=\delta$ against the alternative hypothesis $\mu/\sigma=\delta'$ sequentially, the likelihood ratio test involves a fairly complicated function. The author shows that the logarithm of the likelihood ratio l can be expressed in terms of the solution of a second order differential equation which can be approximated by standard methods thus yielding a much simpler approximating function for $\log l$. The accuracy of the approximation and the effect of studentization on the test are discussed and a numerical example is given illustrating the results of the paper. *R. P. Peterson* (Seattle, Wash.).

Birnbaum, Z. W., and Tingey, Fred H. One-sided confidence contours for probability distribution functions. Ann. Math. Statistics 22, 592–596 (1951).

Let $F(x)$ be the continuous d.f. of a random variable X, and let $F_n(x)$ be the empirical d.f. determined by a random sample $X_1, X_2, \cdots, X_n$. It has been shown by Smirnov that the probability $P_n(\epsilon)$ that $F(x)$ is everywhere $\leqq F_n(x)+\epsilon$ does not depend on $F(x)$. In this paper the authors prove that, for any ϵ such that $0<\epsilon\leqq 1$, $P_n(\epsilon)$ is given by

$$(*)\qquad P_n(\epsilon)=1-\epsilon\sum_{j=0}^{[n(1-\epsilon)]}\binom{n}{j}\left(1-\epsilon-\frac{j}{n}\right)^{n-j}\left(\epsilon+\frac{j}{n}\right)^{j-1}$$

where $[n(1-\epsilon)]$ is the greatest integer contained in $n(1-\epsilon)$. The values $\epsilon_{n,\alpha}$ such that $P_n(\epsilon_{n,\alpha})=1-\alpha$ are tabulated for $\alpha=.10, .05, .01, .001$ and $n=5, 8, 10, 20, 40, 50$. For large n the asymptotic values $\tilde{\epsilon}_{n,\alpha}$ due to Smirnov are given by the formula $\tilde{\epsilon}_{n,\alpha}=[(2n)^{-1}\log\alpha^{-1}]^{\frac{1}{2}}$. For $n=50$, there is very good agreement between $\tilde{\epsilon}_{n,\alpha}$ and $\epsilon_{n,\alpha}$. The error committed by using $\tilde{\epsilon}_{n,\alpha}$ is always in the safe direction if $.001\leqq\alpha\leqq.1$.

B. Epstein (Detroit, Mich.).

Watanabe, Yoshikatsu. Unbiased estimate of the mean absolute deviation. J. Sci. Gakugei Fac. Tokushima Univ. 1, 17–24 (1950).

Let $x_1, \cdots, x_n$ be n independent normal variables with zero means and identical variances. The author finds the conditional expected value of $|x_1|+\cdots+|x_n|$ when the x_i are subject to m homogeneous linear restrictions. For the single restriction $x_1+\cdots+x_n=0$ this leads to the unbiased estimate $\sum_{i=1}^{n}|x_i|/[n(n-1)]^{\frac{1}{2}}$ of the population mean deviation. The method used is that of A. T. Craig [Bull. Amer. Math. Soc. 42, 670–674 (1936)] in solving the corresponding problem for the sample variance. *G. E. Noether.*

Chand, Uttam. Test criteria for hypotheses of symmetry of a regression matrix. Ann. Math. Statistics 22, 513–522 (1951).

Let Y_{ij} ($i=1, \cdots, p$; $j=1, \cdots, N$) be random variables such that the $Y_{ij}-E(Y_{ij})$ constitute a sample from a p-variate normal distribution with unknown covariance matrix and such that $E(Y_{ij})=\sum\beta_{ik}x_{kj}$ where the regression coefficients β_{ik} are also unknown. The author proposes a test statistic which provides a similar region for testing the symmetry of the regression matrix. For the case $p=2$ he derives the distribution of this statistic and shows the resultant test to be unbiased. *E. L. Lehmann.*

Chapman, Douglas G., and Robbins, Herbert. Minimum variance estimation without regularity assumptions. Ann. Math. Statistics 22, 581–586 (1951).

A lower bound is derived for the variance of estimators ". . . which is (a) free from regularity assumptions and (b) at least equal to and in some cases greater than that given by the Cramér-Rao inequality" (see (32.3.3a), page 480, in H. Cramér, Mathematical Methods of Statistics [Princeton University Press, 1946; these Rev. 8, 39]).

D. F. Votaw, Jr. (New Haven, Conn.).

***Wald, A. Asymptotic minimax solutions of sequential point estimation problems.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 1–11. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

Let θ be the unknown parameter which characterizes the density function $f(x|\theta)$ of a chance variable X. Consider the problem of estimating θ sequentially when each observation costs c and the loss function is $(t-\theta)^2$ when t is the estimate. As usual, the risk function $r(\theta, T, c)$ of the procedure T at θ is the sum of the expected value of the loss function and c times the expected value of the number of observations. T_c^0 is called an asymptotic minimax solution if

$$\lim_{c\to 0}\frac{\sup_\theta r(\theta, T_c^0, c)}{\inf_T \sup_\theta r(\theta, T, c)}=1.$$

Let

$$d(\theta)=E\left\{\left(\frac{\partial\log f(x|\theta)}{\partial\theta}\right)^2\middle|\theta\right\},\quad d_0=\inf_\theta d(\theta),$$

and N_c be the smallest integer $\geqq 1/(cd_0)^{\frac{1}{2}}$. The author proves that under certain regularity conditions the following two estimation procedures are asymptotic minimax solutions. 1) Take N_c observations and estimate θ by the maximum likelihood estimate $\hat{\theta}_{N_c}$. 2) Stop taking observations at the smallest positive integer n for which

$$\frac{1}{nd(\hat{\theta}_n)}-\frac{1}{(n+1)d(\hat{\theta}_n)}\leqq c$$

and estimate θ by $\hat{\theta}_n$. Here $\hat{\theta}_n$ denotes the maximum likelihood estimate of θ from the first n observations.

J. Wolfowitz (Ithaca, N. Y.).

Walter, Edward. Über einige nichtparametrische Testverfahren. Mitteilungsblatt Math. Statist. **3**, 31–44 (1951).

Walter, Edward. Über einige nichtparametrische Testverfahren. II. Mitteilungsblatt Math. Statist. **3**, 73–92 (1951).

Expository paper. *W. Hoeffding* (Chapel Hill, N. C.).

Blanc, Charles. Évaluation stochastique de l'erreur dans les formules d'interpolation. C. R. Acad. Sci. Paris **233**, 683–684 (1951).

If $g(x)$ denotes the error in a (linear, parabolic) interpolation the author proposes to consider $m[g(x)]$ and $m[g^2(x)]$ where $m[f(x)]=\lim_{X\to\infty}(2X)^{-1}\int_{-X}^{X}f(x)dx$. *K. L. Chung.*

Blanc, Charles. Évaluation stochastique de l'erreur dans les formules d'intégration numérique. C. R. Acad. Sci. Paris **233**, 726–727 (1951).

Application of the idea in the note reviewed above to numerical integration. *K. L. Chung* (Ithaca, N. Y.).

***Wold, Herman O. A. Series cronologicas estacionarias. [Stationary Time Series].** Monografias de Ciencia Moderna, no. 28. Consejo Superior de Investigaciones Cientificas, Madrid, 1951. 75 pp.

An exposition of recent work on time series as given by the author in lectures at Madrid (1949) and at the University of Lucknow (1950).

Kimball, A. W. On dependent tests of significance in the analysis of variance. Ann. Math. Statistics **22**, 600–602 (1951).

This note discusses an analysis of variance model in which several hypotheses are tested in one experiment using the same error variance in each test. Thus the statistics used for these tests are stochastically dependent. It is shown that the dependence of the statistics tends to increase the probability of making no errors of the first kind. *E. Lukacs.*

Nair, K. R. Some two-replicate partially balanced designs. Calcutta Statist. Assoc. Bull. **3**, 174–176 (1951).

The following series of partially balanced incomplete block designs is constructed: $v=\frac{1}{2}p(p-1)q$, $k=(p-1)q$, $r=2$, $b=p$, $\lambda_1=2$, $\lambda_2=1$, $\lambda_3=0$; $n_1=(q-1)$, $n_2=2(p-2)q$, $n_3=\frac{1}{2}(p-2)(p-3)q$. Also, $v=12$, $k=4$, $r=2$, $b=6$, $\lambda_1=1$, $\lambda_2=1$, $\lambda_3=0$, $\lambda_4=0$, $n_1=4$, $n_2=2$, $n_3=4$, $n_4=1$. The efficiency factors of these designs are also given.

H. B. Mann (Columbus, Ohio).

Nandi, H. K. On the efficiency of experimental designs. Calcutta Statist. Assoc. Bull. **3**, 167–171 (1951).

The situation considered here is as follows: A random variable y is normally distributed with $E(y)=\beta_1x_1+\cdots+\beta_px_p$ where the value of the x_i can be controlled by the experimental design. The hypothesis $\beta_1=\cdots=\beta_k=0$ is to be tested by n independent experiments. Denoting by $y_\alpha, x_{1\alpha}, \cdots, x_{n\alpha}$ the values of y and the x_i in the αth experiment we put $(x_{i\alpha})(x_{j\alpha})=(d_{ij})$ and $(d_{ij})^{-1}=(c_{ij})$. The effect β_i is called orthogonal to β_j if $(x_{i1}, \cdots, x_{in})$ is orthogonal to $(x_{j1}, \cdots, x_{jn})$. If the smallest possible positive value of $|c_{ij}|$ $(i, j=1, \cdots, k)$ is denoted by c_0, then the efficiency of the design is defined by $|c_0|/\|c_{ij}\|$. This definition is due to A. Wald. Let $D_1=(d_{ij})$ $(i, j=1, \cdots, k)$, $C_1=(D_1)^{-1}$. The author proves: If D_1 is given then $\|C_1\|$ will be minimized if $\beta_1, \cdots, \beta_k$ are orthogonal to $\beta_{k+1}, \cdots, \beta_p$. The most efficient r-way design for testing main effects of the first classification is one in which all classes of the first classification contain the same number of observations. This also shows that all orthogonal designs like Latin-squares, etc., have maximum efficiency in the sense of Wald. *H. B. Mann* (Columbus, Ohio).

Mathematical Economics

Arrow, Kenneth J., Harris, Theodore, and Marschak, Jacob. Optimal inventory policy. Econometrica **19**, 250–272 (1951).

Suppose a firm or nonprofit agency is managing an inventory of finished goods. It derives utility from meeting demands for its product. Against this is balanced the cost of ordering and storing goods. In addition, the agency suffers a "depletion penalty" whenever it is unable to meet a demand. The problem of optimal inventory policy is the choice of the buying policy which maximizes the net utility or net expected utility, if random elements enter. Simple (linear or constant) forms for utility and cost functions, and the depletion penalty, are chosen and a sequence of problems of increasing generality is solved. In the very simplest case the demand per unit of time is known and constant. A complete solution for this case is given, specifying the optimal interval between orders and the best size of order. The next problem discussed permits the demand in a period of time to be a chance quantity with known distributions. The best stock at the beginning of the period is one which equates marginal cost (including carrying costs) and marginal expected utility. The standard second-order condition is given.

These problems are static. Optimization over time is considered next. The distribution of demand is assumed the same in all periods; and the marginal utility of an operation is assumed equal to the purchasing price. In addition, only the 2-parameter family of ordering policies is considered which may be characterized as follows: if stock on hand at the beginning of a period is larger than a certain constant s, do not order; if less than s, order enough to bring the stock up to another constant level S. Of course, many other kinds of policies are conceivable, and indeed may be superior to any (s, S) policy. Under these restrictions, and within a given discount factor, a function $L(y_t)$ is defined and interpreted as the present value at time t of all future expected losses, given that the initial stock is y_t. The function $L(y)$ is shown to satisfy the integral equation of renewal theory. The standard solution in terms of an infinite series of convolutions is written down: the resulting $L(y_0)$ is a function of s and S, and may be minimized with respect to these parameters. An example is worked out in which the distribution of demand is of gamma-type, where convolutions can be written explicitly. *R. M. Solow.*

Metzler, Lloyd A. A multiple-region theory of income and trade. Econometrica **18**, 329–354 (1950).

This paper deals with the impact of a change in investment in one region or country upon income in all sectors of an n-region system. The model assumes that all prices, costs and exchange rates are constant, and is specified by a system

of n equilibrium equations:

$$(1)\quad y_i=[u_i(y_i)-m_i(y_i)]+\sum_{j=1}^{n}m_{ij}(y_i) \qquad (i,j=1,2,\cdots,n;\ i\neq j).$$

Here y_i is the national income of the ith country, $m_i(y_i)$ the total import function, and $m_{ji}(y_i)$ represents the imports of the ith region from the jth region expressed as a function of y_i such that $m_i(y_i)=\sum_{j=1}^{n}m_{ji}(y_i)$. The function $u_i(y_i)$ relates both expenditures on consumption and net investment to income y_i. The n equations (1), with given prices and exchange rates are sufficient to determine the level of income in each region.

Consider first the question of the dynamic stability of the system. It is assumed that output increases whenever demand exceeds current output and that the speed of adjustment is proportional to the size of excess demand. The corresponding system of dynamic equations is given by

$$(2)\quad \frac{dy_i}{dt}=k_i\left[u_i(y_i)-m_i(y_i)+\sum_{j=1}^{n}m_{ij}(y_j)-y_i\right], \qquad i\neq j,\ k_i>0.$$

The solution of (2) is given in terms of a linear approximation which yields necessary conditions for stability. A Taylor expansion about the equilibrium values y_i^0 yields

$$(3)\quad \frac{dy_i}{dt}=k_i\left[(u'_i-m'_i-1)(y_1-y_0)+\sum_{j=1}^{n}m'_{ji}(y_j-y_j^0)\right], \quad i\neq j.$$

The coefficients m'_{ij} and u'_j represent the marginal propensity of the jth country to import from the ith country and the marginal propensity of the jth country to spend respectively. (3) may be solved for any given initial conditions such that

$$(4)\quad y_i(t)=y_i^0+\sum_{j=1}^{n}A_{ij}e^{\lambda_j t}$$

where the A_{ij} are constants determined by the initial level of income at $t=0$, and the λ_j are latent roots of the characteristic equation of the system. $y_i(t)$ will approach $y_i^0(t)$, as t increases, only if the real parts of λ_j are negative. The necessary and sufficient conditions for this are that the determinant

$$(5)\quad M\equiv\begin{vmatrix} 1+m'_1-u'_1 & -m'_{12} & \cdots & -m'_{1n} \\ -m'_{21} & 1+m'_2-u'_2 & \cdots & -m'_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ -m'_{n1} & \cdot & \cdots & 1+m'_n-u'_n \end{vmatrix}$$

and any set of its principal minors must be positive. The stability of (3) is thus independent of k_i. These stability conditions depend on u'_i. If $u'_i<1$ in every region, the system must be stable, but if $u'_i>1$ in every region, the system must be unstable. In the intermediate case where some $u'_i>1$, the system may or may not be stable.

Consider now a disturbance of equilibrium due to an increase in investment in Country 1. The first equation of (1) may be rewritten:

$$(1.1')\quad y_1=[u_1(y_1)-m_1(y_1)]+\sum_{j=2}^{n}m_{ij}(y_j)+\alpha_1,$$

where α_1 is autonomous new investment. The remaining $(n-1)$ equations of (1) are unaltered. Solving for $dy_i/d\alpha_1$ we have (6) $dy_i/d\alpha_1=M_{1i}/M$, where M is given by (5). For a stable system, M and $M_{1i}>0$, hence $dy_i/d\alpha_1>0$. $dy_1/d\alpha_1$ is a generalized investment multiplier, and for the normal case $(u'_i<1)$ the limits $1/(1-u'_1+m'_1)<dy_1/d\alpha_1<1/(1-u'_1)$, hold, where $1/(1-u'_1+m'_1)$ is the foreign trade multiplier which ignores the effects of income changes in other countries upon the given country's exports and $1/(1-u'_1)$ is the ordinary investment multiplier which ignores foreign trade leakages. If all $u'_1\leqslant 1$, the upper limit will not necessarily hold.

Certain generalizations concerning each country's balance of payments are possible. Let b_k denote the balance of payments of country k. From (1) follow

$$(7)\quad b_k=y_k-u_k(y_k), \quad k=2,3,\cdots,n,$$

$$(8)\quad \frac{db_k}{d\alpha_1}=(1-u'_k)\frac{dy_k}{d\alpha_1}.$$

Since in a stable system $dy_k/d\alpha_1>0$, (8) shows that the direction of change of Country k's balance of payments depends on u'_k. It follows that if all $u'_k<1$, the balance of payments must move against Country 1 and in favor of all other countries in the system. If some of the $u'_k>1$, no prediction is possible without additional information. For Country 1, (9) $b_1=y_1-u_1(y_1)-\alpha_1$, and (10) $db_1/d\alpha_1=(1-u'_1)dy_1/d\alpha_1-1$. For the normal case, $dy_1/d\alpha_1<1/(1-u'_1)$, and hence (11) $-1<db_1/d\alpha_1<0$. But if $u'_1>1$, the system is unstable and the limits in (11) no longer hold. There follows a comparison of the two-country and the multiple-country models.

M. P. Stolts (Providence, R. I.).

Chipman, John S. The multi-sector multiplier. Econometrica **18**, 355–374 (1950).

In this paper a multiplier applicable to a multi-sector economy is formulated [see the preceding review]. For an n-sector economy let Y_i denote the income of sector i, E_i the expenditure in sector i, and ${}_jE_i'=\partial E_i/\partial Y_j$ the marginal propensity of sector j to spend in sector i. The ultimate increments in income in sector i, dY_i, resulting from autonomous increments in expenditure, dE_i, may be expressed by:

$$(1)\quad dY_i=dE_i+\sum_{j=1}^{n}{}_jE_i'dY_j \qquad (i=1,2,\cdots,n).$$

In matrix form this becomes: (2) $(I-M)dY=dE$ where I is the unit matrix, $M=[{}_jE_i']$, $dY=\{dY_i\}$ and $dE=\{dE_i\}$ $(i,j=1,2,\cdots,n)$. The multiplier may then be written: (3) $dY=(I-M)^{-1}dE$. Let Δ denote the determinant of $(I-M)$, and $d\Delta_i$ the determinant of the matrix formed by substituting dE in the ith column of $(I-M)$; then (4) $dY_i=d\Delta_i/\Delta$, and the total income generated by dE in the whole economy is (5) $\sum_{i=1}^{n}dY_i=\sum_{i=1}^{n}d\Delta_i/\Delta$. Put $dE_i=e_i dA$, where dA is a disturbance in expenditure. Then $d\Delta_i/dA=\Delta_i'$, where Δ_i' is formed by substituting $e=\{e_i\}$ for dE in $d\Delta_i$. The multiplier for the ith sector is (6) $dY_i/dA=\Delta_i/\Delta$, and for the entire economy

$$(7)\quad \sum_{i=1}^{n}\frac{dY_i}{dA}=\frac{\sum_{i=1}^{n}\Delta_i}{\Delta}.$$

By assigning different forms to e the effects of various kinds of expenditure disturbances may be analyzed. It is also shown that if the marginal propensities to spend are equal in all sectors, the multi-sector multiplier reduces to the conventional multiplier. The dynamic multiplier is developed on the assumption that dY_i at time t is a function of the income of all sectors in the period $(t-1)$. Hence (1) may be written:

$$(8)\quad dY_i(t)=dE_i+\sum_{j=1}^{n}{}_jE_i'dY_j(t-1) \qquad (i=1,2,\cdots,n)$$

or, (9) $IdY(t)-MdY(t-1)=dE$. The general solution, by standard methods, is given by

$$dY_i=\frac{d\Delta_i}{\Delta}+\sum_{j=1}^{n}k_{ij}\lambda_j{}^t, \tag{10}$$

and

$$\sum_{i=1}^{n}dY_i=\frac{\sum_{i=1}^{n}\Delta_i}{\Delta}+\sum_{i=1}^{n}\sum_{j=1}^{n}k_{ij}\lambda_j{}^t \tag{11}$$

where k_{ij} are arbitrary constants and λ_j are the roots of the characteristic polynomial $|I\lambda-M|=0$. The static multiplier is seen to be a special case of the dynamic multiplier. Necessary and sufficient conditions for stability of the dynamic system are given in terms of the Schur conditions that the roots of the characteristic polynomial lie within the unit circle of the complex plane.

Certain relationships between the static and dynamic multipliers are developed in terms of necessary conditions for the stability of the latter. The essential results are: (1) A necessary condition for stability is that the absolute value of the average marginal propensity to spend internally must be less than unity; (2) a necessary condition for stability is that $|I-M|>0$; (3) if all dE_i are of the same sign, and all ${}_jE_i'>0$, $i\neq j$, the multi-sector multiplier is greater than $\frac{1}{2}$; (4) if all ${}_jE_i'>0$, $i\neq j$, a necessary condition for stability is that each ${}_jE_i'<1$, $i=j$. There follows a section on the effect of changes in relative prices. *M. P. Stoltz.*

Wald, Abraham. On some systems of equations of mathematical economics. Econometrica 19, 368–403 (1951). Translated from Z. Nationalökonomie 7, 637–670 (1936).

Guiraum, A., Tena, J., and Wold, H. Cartes d'indifférence à fonctions de demande données. Trabajos Estadistica 1, 49–68 (1950). (French. Spanish summary)

In regular problems of consumer's theory, one may reverse the usual procedure and from empirically observed functions of n goods demanded in relation to all prices and income deduce the implied indifference surfaces expressed in terms of quantities. The symmetry and the negative semi-definiteness of the matrix of substitution terms assures that the n implicit partial differential equations for the indifference slopes are soluble, but no explicit solution is generally possible. Even in the two-good case, some limitation on the demand functions is needed to reach an explicit solution, since use of the homogeneity properties still leaves us with two implicit equations. The authors consider three special cases where the demand for one of the goods is (1) independent of its own price, (2) independent of the other price, and (3) independent of income. Whenever an extra partial differential identity of this type is imposed on the problem, there remains but one arbitrary function and it is shown that the indifference curves of the respective cases have the properties: (1) the difference between any two indifference curves, as measured in terms of the other good, is proportional to the quantity of the good in question; (2) the difference between any two indifference curves, as measured in terms of the good itself, is proportional to its quantity; (3) the difference between any two indifference curves, as measured in terms of the other good, is strictly constant, as in the classical case of constant marginal utility of the other good. Cases (1) and (2) are necessarily of regular convexity and slope in only some limited region, as the graphs show. A fourth case is that in which the income-consumption curves are known to be straight lines passing through a given indifference curve with specified slopes. It is mentioned that straight-line price-consumption curves lead to implicit differential equations. Finally, the special Törnqvist [Ekonomisk Tidskrift 43, 216–225 (1941)] family of income consumption curves, whose consumption is given by certain simple rational functions of income with parameters depending in a simple way on prices, is integrated by elementary integrals.

P. A. Samuelson (Cambridge, Mass.).

Cansado, E., and Wold, H. Some properties of price-consumption curves and income-consumption curves. Trabajos Estadistica 1, 37–48 (1950).

Through a given consumption point n curves can be passed by varying in turn each of n single prices. It is shown that these form a "corner," containing inside of them the curve generated by a pure income change. A new set of n curves is defined by changing all but a single price in the same proportions. For a utility surface linear in the logarithms of three goods, it is proved that each of the new curves is exactly perpendicular to the corresponding member of the previous family; and it is stated that typical empirical cases will approximate to perpendicularity. It is proved that the new set of curves also contain inside themselves the income-consumption curve. Illustrating with geometrical argument, the authors define substitution vectors for the second set of curves analogous to the familiar substitution vectors for the first set and prove for both an interesting conjugate property, utilizing for this purpose the property of quadratic forms negative definite under constraints.

P. A. Samuelson (Cambridge, Mass.).

Cansado, E. Vector interpretation of Slutsky's equation. Trabajos Estadistica 1, 29–36 (1950). (Spanish and English)

The fundamental fact in consumption theory, that a pure price change can be decomposed into an income-effect and substitution-effect, is given a vector interpretation. A graph for the two-good case illustrates that the order of the resolution is immaterial for first-order changes.

P. A. Samuelson (Cambridge, Mass.).

Scott, A. D. Bibliography of applications of mathematical statistics to economics, 1943–1949. J. Roy. Statist. Soc. Ser. A. 114, 372–393 (1951).

Mathematical Biology

***Rashevsky, Nicolas. Mathematical Biology of Social Behavior.** The University of Chicago Press, Chicago, Ill., 1951. xii+256 pp. $5.00.

Social phenomena, A, are statistical consequences of individual behavior, B, which in turn depends on brain physiology, C. So if we knew all about C we might in principle deduce all about A. Or starting from postulates consistent with the known facts regarding C, and further postulates if necessary at the intervening levels, we may work out the mathematical consequences of our model or chain of models, checking numerical values of the parameters assumed and the quantitative predictions against experimental psychological and historical sociological investigations. The aim will be first accurate description and then prediction; and it is to be expected that we must proceed rather tentatively at first, trying out many alternative models. That roughly

is the author's plan for mathematical sociology. His earlier books (1) Mathematical Biophysics [Univ. of Chicago Press, Chicago, 1938, revised 1948], (2) Mathematical Theory of Human Relations [Principia Press, Bloomington, Ind., 1947] were confined to the intervals (C, B), (B, A); the present work aims at covering the whole stretch, continuing (1) and (2) as well as deriving the postulates of (2) from biophysical premises. Part I, "Brain Mechanism and Behavior", is mostly a condensed version of the relevant parts of (1). Part II, dealing with the distribution of wealth and the formation of social classes, unfortunately has not much connection with Part I and hence seems rather out of its logical order in the scheme. Parts III, IV, V discuss behavior (of individuals and social groups), classified under the three not rigorously exclusive headings of Imitative, Motivational (or hedonistic) and Learned, with numerous subheadings. Part VI gathers together the chaotic-seeming threads of the preceding parts by using them all in some models of "socioeconomic dynamics", where a special relevance to contemporary events is intended.

There is much that can be said against the book. Some of it the author anticipates in his very modest preface, and the reviewer is tempted to put it even more strongly: no approximating device seems too crude or inelegant, no assumption concerning human behavior too oversimplified, naive or unreal to be employed as the author heroically hacks his way (that is the impression one gets) from C to A. The dependence of the sociology on the physical biology is at important places absent or not so close as is made to appear. With few exceptions the stage of actual quantitative verification is never reached. But to end the criticism here would be altogether unjust. It must be recognised that the author's immediate aim, and indeed his achievement, is not to prove mathematically anything particular about human behavior, but rather to demonstrate the possibility in principle of such a branch of applied mathematics. His reiterated phrase "this formula is in principle verifiable" may be read "in a more satisfactory model some corresponding formula would be obtainable and verifiable". And if and when some more satisfactory mathematical social science arrives, many of its features may well be found to have had their crude counterparts here. In any case this book deserves the attention of all workers in this field; it is likely to prove stimulating and influential, both for the great variety of its ideas and for the resourcefulness of its manipulative technique. It is very well produced although there are a fair number of misprints; but three badly needed concessions to the reader are lacking: convenient summaries, an index of notations, and (since there are 27 chapters with independently numbered equations and very frequent cross references) chapter numbers in page headings. *I. M. H. Etherington* (Edinburgh).

Peyovitch, T. Équations différentielles biologiques. Bull. Soc. Math. Phys. Serbie **2**, nos. 3–4, 9–25 (1950). (Serbo-Croatian. French summary)

A study by the method of successive approximation of the "logistic equations" $x_i' = x_i(e_i - \sum l_{ik}x_k)$ in the neighborhood of the equilibrium other than the origin, supposing it to exist uniquely. *A. S. Householder.*

Shimbel, Alfonso. Applications of matrix algebra to communication nets. Bull. Math. Biophys. **13**, 165–178 (1951).

For a communication system relating n message sources, whose interconnections are described by a structure matrix and whose messages consist solely of descriptions of this interconnection (spreading from source to source with each communication), the author shows that the matrix describing the status of knowledge of interconnections after k communications is the kth power of the structure matrix, or as an immediate generalization, the product of the k structure matrices for each communication when these are variable. Several measures of the efficiency of such a system in spreading knowledge are considered. *J. Riordan.*

TOPOLOGY

Ungar, Peter. A theorem on planar graphs. J. London Math. Soc. **26**, 256–262 (1951).

If I is a set of vertices of a finite graph G, let $G(I)$ be the graph obtained from G by suppressing the members of I and their incident edges. If k is any positive integer >1, let $n_k(G)$ be the least integer such that for some I of $n_k(G)$ members the components of $G(I)$ have at most k vertices each. The author's main theorem states that

$$n_k(G) < 12k^{-1}(\log k)^{-1}\alpha_0(G)$$

if G is planar, where $\alpha_0(G)$ is the number of vertices of G. He remarks that this inequality is non-trivial only if $k > 10^5$. The result does not extend to non-planar graphs; the author shows that for a given k we can find arbitrarily large non-planar graphs G such that $n_k(G) > \frac{1}{4}\alpha_0(G)$. *W. T. Tutte.*

Müller, Gert H. Über die Eigenschaften der Teilmengen eines Kuratowski'schen Raumes. II. Portugaliae Math. **10**, 53–70 (1951).

In this part [cf. same journal **9**, 149–167 (1950); these Rev. **12**, 434] the author restricts his postulate system by requiring that $(X+Y)^- = X^- + Y^-$. He also considers the relative topology of subsets. Eight types of connectedness are defined and these are characterized in Satz 8, in terms of indecomposability properties. Thus a space is connected (in the usual sense) if it is indecomposable relative to the property of being non-void and closed—in other words, is not the union of a pair of non-void disjoint closed sets. In part (a) of Satz 8 are given 11 properties equivalent with 0-connectedness. The remainder of the paper is essentially an explication of the results given here. It is not feasible to report in detail on the paper, which also depends strongly on Part I. *A. D. Wallace* (New Orleans, La.).

Smirnov, Yu. M. On normally disposed sets of normal spaces. Mat. Sbornik N.S. **29(71)**, 173–176 (1951). (Russian)

Let R be a completely regular topological space. A subset M of R is said to be normally imbedded in R if for every open set G such that $M \subset G \subset R$, there exists a set H of type F_σ such that $M \subset H \subset G$. The principal results proved are as follows. Every set normally imbedded in a normal space is normal in its relative topology. Let a topological space be called finally compact if every open covering of that space admits a countable subcovering. If a completely regular space is finally compact, then it is normally imbedded in every completely regular space containing it. A completely regular space R is finally compact if and only if it is normally imbedded in βR. If R is completely regular and if M is normally imbedded in R, then $\dim M \leqq \dim R$

(dim denotes the dimension defined by finite open coverings). If R_1 and R_2 are regular spaces and if $R_1 \times R_2$ is finally compact, then R_1, R_2, and $R_1 \times R_2$ are normal and $\dim(R_1 \times R_2) \leqq \dim R_1 + \dim R_2$. *E. Hewitt* (Uppsala).

Smirnov, Yu. M. Some relations in the theory of dimensions. Mat. Sbornik N.S. **29(71)**, 157–172 (1951). (Russian)

Let R be a topological space. Let $\dim R$ be the dimension of R defined in terms of open coverings. Let $\operatorname{ind} R$ be the dimension of R defined in the usual inductive fashion [see, for example, Hurewicz and Wallman, Dimension Theory, Princeton Univ. Press, 1941; these Rev. **3**, 312]. Let $\operatorname{Ind} R$ be the dimension of R, again defined inductively, but where arbitrary closed sets F and arbitrary open sets containing F replace the points and open neighborhoods which occur in the definition of $\operatorname{ind} R$. It is known that $\dim R = \operatorname{ind} R = \operatorname{Ind} R$ for R separable and metric [Hurewicz and Wallman, op. cit.]; that $\dim R \leqq \operatorname{ind} R$ for R a bicompact Hausdorff space [P. Aleksandrov, Soobščeniya Akad. Nauk Gruzinskoĭ SSR **2**, 1–6 (1941); these Rev. **3**, 58]; that $\dim R \leqq \operatorname{Ind} R$ for R normal [Vedenisov, Izvestiya Akad. Nauk SSSR. Ser. Mat. **5**, 211–216 (1941); these Rev. **3**, 58]; that there exists a bicompact Hausdorff space T for which $\dim T = 1$, $\operatorname{ind} T = 2$ [Lunc, Doklady Akad. Nauk SSSR (N.S.) **66**, 801–803 (1949); these Rev. **11**, 46]. The present paper is concerned with extending these inequalities, where possible, and with other dimension-theoretic problems. The following theorems are proved. (I) Let R be a normal space for which every open covering admits a countable subcovering. Then $\dim R \leqq \operatorname{ind} R$. (II) There exists a normal space S for which $\operatorname{ind} S < \operatorname{Ind} S$. This space is a minor modification of Lunc's example [loc. cit.]. (III) Let βR denote the Čech-Stone bicompactification of the (completely regular) space R. Let R be a space with the property that every subspace of R is normal in its relative topology. Then $\operatorname{ind} \beta R = \operatorname{Ind} R = \operatorname{Ind} \beta R$. (IV) For the space S referred to in (II), $\operatorname{ind} S < \operatorname{Ind} \beta S$. (V) Let R be a normal space such that for every subspace U of R, every open covering of U admits a countable subcovering. Then $\operatorname{ind} \beta R = \operatorname{Ind} R = \operatorname{ind} R = \operatorname{Ind} \beta R$. (VI) Let R be a space such that every subspace of R is normal in its relative topology. Then for arbitrary subsets M and N of R, the inequality $\delta(M \cup N) \leqq \delta(M) + \delta(N) + 1$ obtains, where δ is any of the functions ind, Ind, dim. *E. Hewitt.*

Arens, Richard. Operations induced in function classes. Monatsh. Math. **55**, 1–19 (1951).

A study of normed linear algebras, in particular the algebra $C(X, R)$ of continuous functions on a compact Hausdorff space, is made as a special case of systems called phyla. A phylon is a class of sets (A) and a class (F) of functions $f: A \to B$ on $A \varepsilon (A)$ to $B \varepsilon (A)$ subject to the following conditions. If $A \varepsilon (A)$, the identity $A \to A \varepsilon (F)$. If $A, B \varepsilon (A)$, the cartesian product $A \times B \varepsilon (A)$. If A, B, $C \varepsilon (A)$ then $(A \times B) \times C \to A \times (B \times C)$ and $A \times (B \times C) \to (A \times B) \times C$ are in (F) with $((a, b), c)$ and $(a, (b, c))$ paired by each function. If f is $A \to B$ and g is $B \to C$ there is an $h \varepsilon (F)$ such that $h(a) = g(f(a))$ on A. The inner connections between (A) and (F) characteristic of a phylon follow. Let B^A be the set of $A \to B \varepsilon (F)$. If A, $B \varepsilon (A)$, $B^A \varepsilon (A)$. If $m \varepsilon (F)$ is $A \times B \to C$, then there is $n: A \to C^B \varepsilon (F)$ such that $m(a, b) = n(a)(b)$ for $a \varepsilon A$, $b \varepsilon B$. If $f \varepsilon (F)$ is $A \to B$ then for $C \varepsilon (A)$ there is $f': C^B \to C^A \varepsilon (F)$ such that $f'(g)(a) = g(f(a))$ for $g \varepsilon C^B$, $a \varepsilon A$. If A, $B \varepsilon (A)$, $A \times B \to B \times A \varepsilon (F)$.

The main general result may be indicated as follows. For fixed $K \varepsilon (A)$ write B^* for K^B where $B \varepsilon (A)$ and B^{**} for $(B^*)^*$. If m is a binary operation (product) in A, m has an extension to a product μ in A^{**} such that a "natural" mapping $A \to A^{**}$ is a homomorphism. Further if $B \varepsilon (A)$ also has a binary operation and $f: A \to B$ is a homomorphism, then the "natural" extension of f to $A^{**} \to B^{**}$ is also a homomorphism.

Problems arising in the formation of phyla out of topological spaces are studied and some are solved for linear normed spaces. Of interest is the problem of finding conditions under which commutativity for products is transmitted from A to A^{**}. It is shown that this is the case in passing from $C(X, R)$ to $C(X, R)^{**}$. *L. W. Cohen.*

Freudenthal, Hans. Ein Kompaktheitskriterium. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 295–296 (1951).

van Est, W. T., and Freudenthal, Hans. A note on a compactness criterion of H. Freudenthal. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 369–370 (1951).

In the first paper it is proved that if R is a metric space, then the following statements are equivalent: (1) R is compact (= bicompact). (2) If F is an arbitrary set of real continuous functions on R such that $\{f_1, f_2\} \subset F$ implies $\max(f_1, f_2) \varepsilon F$ and $f \varepsilon F$ implies $\inf f = 0$, then there is a 'sequence x_i in R with $\lim f(x_i) = 0$ for each $f \varepsilon F$'. The second paper extends this result by showing that complete regularity of R implies equivalence of (1) and (2). It uses (in addition to the argument of the first paper) the fact that if R is completely regular, then '$\cdots$' may be replaced in (2) by 'point $x \varepsilon R$ with $f(x) = 0$ for each $f \varepsilon F$'. *V. L. Klee, Jr.* (Princeton, N. J.).

White, Paul A. On a certain class of set theoretic properties. Ann. Mat. Pura Appl. (4) **31**, 99–110 (1950).

This paper is concerned with properties $\mathcal{P}$ of a Hausdorff space which (like, e.g., connectedness) satisfy the well-known "Polish theorem" (A): If $H \cup K$ and $H \cap K$ both have $\mathcal{P}$, and if the sets $H - K$, $K - H$ are mutually separated, then H and K both have $\mathcal{P}$. The "local" and "strong local" forms ($l\mathcal{P}$ and $sl\mathcal{P}$) of a property $\mathcal{P}$ are defined by: S is $(s)l\mathcal{P}$ if each point (subset of S having $\mathcal{P}$) has arbitrarily small neighborhoods in S which have $\mathcal{P}$. Theorem 2: If (A) holds for $\mathcal{P}$, then (A) holds for $l\mathcal{P}$ whenever $H \cup K$ has $sl\mathcal{P}$. [Since this assumes that $\mathcal{P}$ is an absolute property, some care is needed in interpreting subsequent results.] Theorem 4 asserts that, for each of a group of 12 familiar properties satisfying (A), the local form implies the strong local form (so that the local forms of these properties also satisfy (A)); unfortunately, as the author states in a communication to the reviewer, the proof of Theorem 4 is incorrect. A second group of 13 fundamental properties is given, each of which satisfies (A), though not locally. Finally, some of the connectivity properties are generalised to higher dimensions (using Čech homology theory), and it is shown that, for compact spaces, and under supplemental hypotheses in some cases, the generalised properties still satisfy (A). There are many misprints. *A. H. Stone.*

Valentine, F. A. A characterization of simply connected closed arcwise convex sets. Proc. Amer. Math. Soc. **2**, 778–780 (1951).

E_2 denotes the plane. A set $S \subset E_2$ is unilaterally connected if for each $x, y \varepsilon S$ there is a continuum M, containing $x \cup y$ and lying in the common part of S and a closed half-plane whose boundary contains $x \cup y$. A set $S \subset E_2$ is arcwise con-

vex if for each $x, y \in S$ there is an arc A from x to y, such that $A \subset S$ and such that A lies in the boundary of its own convex hull. Theorem: A simply connected closed set $S \subset E_2$ is arcwise convex if and only if S is unilaterally connected.
E. E. Moise (Ann Arbor, Mich.).

Swain, R. L. Approximate isometries in bounded spaces. Proc. Amer. Math. Soc. 2, 727–729 (1951).

A transformation of a metric space into another is called an ϵ-isometry if it changes distances by an amount less than ϵ, where ϵ is a given positive number. According to a result of Hyers and Ulam [Bull. Amer. Math. Soc. 51, 288–292 (1945); these Rev. 7, 123], to each isometry of a real Hilbert space onto itself, there corresponds a true isometry such that the distance between the images of any point under the two transformations is less than $k\epsilon$, where $k \leqq 10$. In the present paper it is shown by a simple example that no single value of k can exist for all ϵ-isometries of all bounded subsets of the plane into other bounded subsets. However the following theorem is proved. If M is a subset of a compact metric space S and $\eta > 0$ is given, then there exists a positive number ϵ such that if T is any ϵ-isometry of M into a subset of S, there exists an isometry U of M such that $\rho(T(x), U(x)) < \eta$ for all x in M, where ρ is the distance function. *D. H. Hyers* (Los Angeles, Calif.).

Nagumo, Mitio. Degree of mapping of manifolds based on that of Euclidean open sets. Osaka Math. J. 2, 105–118 (1950).

As used here, the term m-manifold denotes a connected topological space covered by countably many open sets U_i with each of which is associated a homeomorphism φ_i of $\bar{U}_i$ onto the closed euclidean disc $\bar{K}$ of dimension m; and such that no compact set meets more than a finite number of the sets U_i. The first approximation to a general definition of degree of mapping for such manifolds is made for "α-mappings," which (for the open sets concerned) have only countable counter-images for each image point. If M and M' are m-manifolds with systems (U_i, φ_i) and (V_j, ψ_j), respectively, and f is an α-mapping of an open subset D of M (such that $\bar{D}$ is compact) into M', then a degree $A(a, D, f)$ is defined for $a \in M'$ by means of the (as usually defined) degrees $A(\psi(a), \varphi(G_\nu), \psi f \varphi^{-1})$, where the G_ν are open sets, finite in number, covering $f^{-1}(a)$ and small enough to lie in sets $D \cap U_{i(\nu)}$, etc. For the general mapping f, it is then possible to define a degree $A(a, D, f)$ in terms of the degree $A(a, D, f^*)$ of an approximating α-mapping f^*; for sufficiently "close" approximations, the degree is unique. The existence of approximating α-mappings, and the desired justification theorems (establishing properties analogous to those possessed by the degree of mappings in euclidean space) are established. *R. L. Wilder.*

Dugundji, J. An extension of Tietze's theorem. Pacific J. Math. 1, 353–367 (1951).

Tietze's theorem says that any mapping into the 1-dimensional Euclidean space E^1 of a closed subset of a metric space can be extended to a mapping of the whole space. It is trivial that E^1 can be replaced by E^n or the Hilbert cube. The main theorem of the present paper extends Tietze's theorem to the case of mappings into any locally convex linear space. Several consequences of this theorem are derived. One of these is that the unit sphere of a normed linear space has the fixed point property if and only if it is compact. A second application is to the theory of absolute neighborhood retracts and locally connected spaces. It is shown here that this theory, previously developed only for separable metric spaces, does in fact hold for arbitrary metric spaces. *E. G. Begle* (New Haven, Conn.).

Gustin, W., and Sullivan, J. A. Contractions in a hyperbolic space. Duke Math. J. 18, 665–671 (1951).

Le but de ce travail est d'étendre aux "espaces hyperboliques" le théorème de Wolff sur l'existence d'un point fixe dans une transformation holomorphe d'un cercle en une partie de lui même. Un espace hyperbolique est un espace métrique compactifié par adjonction de points idéaux. Des axiomes concernant le convergence des points ordinaires vers ces points idéaux, ainsi que la "fonction de comparison métrique", définissent l'espace hyperbolique dont un modèle est fourni par l'image conforme bien connue de la géométrie plane hyperbolique. Un théorème connu de Julia sur la contraction par les transformations holomorphes bornées [Acta Math. 42, 349–355 (1920)], convenablement adapté, sert à établir le théorème de Wolff [Math. Z. 26, 125–127 (1927)] dans les espaces hyperboliques. *S. Stoïlow.*

Majstrenko, Petro. A set-theoretic generalization of the principle of analytic continuation. I. Proc. London Math. Soc. (3) 1, 152–162 (1951).

Starting with a set ϕ of distinct single-valued transformations of subsets of a topological space X into a topological space Y which are called elements of ϕ, the author indicates by an extended sequence of definitions a method of constructing multiple-valued functions in X to Y from these elements which are called "functions in the large belonging to ϕ." This comprises a set-theoretic topological analog and generalization of the principle of analytic continuation and of the concept of a monogenic analytic function. Topological intersection and chainability of the graphs of the elements of ϕ is the basic idea involved and this replaces the power series developments of the Weierstrass process of analytic continuation. A similar generalization of the concept of a Riemann surface of an analytic function and of a Weierstrass "analytisches Gebilde" is obtained.
G. T. Whyburn (Charlottesville, Va.).

Radó, Tibor. On identifications in singular homology theory. Rivista Mat. Univ. Parma 2, 3–18 (1951).
Radó, Tibor. An approach to singular homology theory. Pacific J. Math. 1, 265–290 (1951).

Let X be a topological space and H a real separable Hilbert space. The author defines a complex $R(X)$ with q-dimensional cells $(v_0, \cdots, v_q, T)$, where the v_i are points of H and T is a continuous function on the convex hull of $(v_0, \cdots, v_q)$ to X. With the natural incidence relations, $R(X)$ becomes a closure finite complex which is, in a certain sense, the "largest" singular complex which has been studied. The principal result of the first paper is that the integral homology groups of $R(X)$ are isomorphic with the usual singular homology groups of X.

The second paper is an intensive study of identifications in $R(X)$, where "identification" has the following pleasantly precise meaning. For each non-negative integer p let C_p be the integral chain group of $R(X)$, so that $\{C_p\}$ is a Mayer complex (or chain complex). If $\{E_p\}$ is a sub Mayer complex of $\{C_p\}$ such that the natural homomorphism of the homology groups of $\{C_p\}$ into those of the quotient complex $\{C_p/E_p\}$ is, for each p, an isomorphism onto, then $\{E_p\}$ is an inessential identifier. Let T_p be the subgroup of C_p generated by chains of the form

$$(v_0, \cdots, v_i, v_{i+1}, \cdots, v_p, T) - (v_0, \cdots, v_{i+1}, v_i, \cdots, v_p, T),$$

and let D_p be generated by chains $(v_0, \cdots, v_p, T)$ such that $v_j = v_{j+1}$ for some j. Let A_p be generated by chains $(v_0, \cdots, v_p, T) - (w_0, \cdots, w_p, T\alpha)$ where the w's form a linearly independent set and α is an affine map such that $\alpha(w_i) = v_i$ for each i. The principal theorem: Each of $\{A_p\}$, $\{D_p\}$, $\{T_p\}$, $\{A_p+D_p\}$, $\{A_p+T_p\}$, $\{D_p+T_p\}$, $\{A_p+D_p+T_p\}$ and {the division hull in C_p of $A_p+D_p+T_p$} is an inessential identifier. The theorem is a precise statement of the effect of identifying (roughly) affinely equivalent cells, degenerate cells, transposed cells or combinations of these. Speaking somewhat imprecisely, the theorem extends earlier theorems; notably, one of type $\{D_p\}$ of A. W. Tucker [Mat. Sbornik N.S. 3(45), 287–289 (1938)] and one of type $\{T_p\}$ of S. Eilenberg [Ann. of Math. (2) 45, 407–447 (1944); these Rev. 6, 96]. [Note: The lemmas on Mayer complexes which the author reproves after citing unpublished work are available in a paper of J. L. Kelley and E. Pitcher [ibid. (2) 48, 682–709 (1947); these Rev. 9, 52].] *J. L. Kelley.*

Kudô, Tatsuji. The homological construction of fibre bundles. Sûgaku (Mathematics) 3, 12–28 (1951). (Japanese)

A comprehensive report covering different aspects of the homology theory of fibre bundles. *S. Chern.*

Kundert, E. G. Über Schnittflächen in speziellen Faserungen und Felder reeller und komplexer Linienelemente. Ann. of Math. (2) 54, 215–246 (1951).

The author studies the problem of finding cross-sections in fiber bundles with a connected structure group in which the fiber is a compact manifold Φ such that the first non-vanishing homotopy group of Φ is the kth (where $k=1$ or 2) and the second non-vanishing homotopy group of Φ is the $(m-1)$st and is cyclic. Thus, there are integers k, m with $k=1$ or 2 and $m-1>k$ such that $\pi_k(\Phi)\neq 0$, $\pi_{m-1}(\Phi)$ is cyclic, and $\pi_i(\Phi)=0$ for $i<k$ and $k<i<m-1$. Assuming the first obstruction is zero, cross-sections can be constructed over K^{m-1}, the $(m-1)$-dimensional skeleton of the base K. For every such cross section F there is a cocycle $c^m(F)$ (coefficients in $\pi_{m-1}(\Phi)$ if $k=2$ and coefficients mod 2 if $k=1$) which measures the obstruction to extending F over K^m. If F and G are two cross-sections over K^{m-1}, their difference cocycle $d^k(F, G)$ (coefficients in $\pi_k(\Phi)$) measures the obstruction to deforming F to G over K^k.

Assuming that $\dim \Phi = m-k$ and that if $k=2$, Φ is an orientable manifold and if $k=1$ the cross section F maps the boundary S^{m-1} of any m-simplex of K into Φ with degree 0 mod 2, the author defines a cocycle $\hat{c}^m(F)$ (coefficients in $H_2(\Phi)$ if $k=2$ and in $H_1(\Phi)$ reduced mod 2 if $k=1$) by the following: If $k=2$ and F is a cross-section over K^{m-1}, for any m-cell σ of K, F defines a map $f: S^{m-1} \to \Phi$. The inverse image of a point of Φ is a 1-cycle of S^{m-1}. Let D be a two chain which it bounds. Then $f(D)$ is a 2-cycle in Φ, and we define $(\hat{c}^m(F))(\sigma)$ to be the homology class of $f(D)$. Similarly we define $\hat{c}^m(F)$ if $k=1$. If F and G are two cross sections over K^{m-1}, a cocycle $\phi^{m-k}(F, G)$ (coefficients integers if $k=2$ and integers mod 2 if $k=1$) is defined which measures the obstruction to separating F and G over K^{m-k}. The author states the following formula of H. Hopf [Colloque de topologie, Bruxelles, 1950, pp. 117–121, Thone, Liège, Masson, Paris, 1951; these Rev. 13, 151]

$$(1) \qquad \hat{c}^m(F) - \hat{c}^m(G) \sim d^k(F, G)\phi^{m-k}(F, G)$$

which connects those cocycles.

Of particular interest is the case where the fiber is a projective space of dimension $n-1$. If the space is a real projective space, we have $k=1$ and $m=n$; if it is a complex projective space, $k=2$ and $m=2n$. Such fiberings are called P-fiberings. For a P-fibering, $c^m(F) = \hat{c}^m(F)$ if $m>2$. The main result is the formula

$$(2) \quad \Phi(F, G) \sim d^{n-1}(F, G) + c_n(G)d^{n-2}(F, G) + \cdots + c_i(G)d^{i-2}(F, G) + \cdots + c_2(G),$$

where the upper indices denote powers in the cohomology ring of K. The $c_i(G)$ are cocycles of (real or complex) dimension $n-i+1$ and can be interpreted as obstructions in suitable fiberings which depend on the cross-section G. This formula is proved inductively from Hopf's formula. Relations between $c_i(F)$ and $c_i(G)$ are then found, and from these relations certain invariants Δ_i of the fibering are constructed. Using (1) and (2) the following is obtained

$$(3) \qquad c(F) = d^n(F, G) + c_n(G)d^{n-1}(F, G) + \cdots + c_1(G),$$

which shows how the secondary obstructions $c(F)$ and $c(G) = c_1(G)$ are related.

The author next considers the associated bundles obtained from a P-fibering by taking as the new fiber r orthogonal (unitary orthogonal in the complex case) points of the projective space which is the fiber. The corresponding fiberings are called U-fiberings. A formula similar to (3) is proved for U-fiberings and relations are obtained between the obstructions of a P-fibering and those of its associated U-fiberings.

The last section contains applications of these results to the question of the existence of fields of complex line elements on a complex manifold. For example, the Chern characteristic classes of a complex manifold occur as the coefficients in the analog of (3) for U-fiberings in this case. *E. H. Spanier* (Chicago, Ill.).

Chang, S. C., and Whitehead, J. H. C. Note on cohomology systems. Quart. J. Math., Oxford Ser. (2) 2, 167–174 (1951).

Given a finite connected polyhedron P such that $\dim P \leqq n+2$ and $\pi_r(P)=0$ for $r \leqq n-1$, Chang [Proc. Roy. Soc. London. Ser. A. 202, 253–263 (1950); these Rev. 12, 120] introduced new numerical invariants of the polyhedron P. These invariants, called the secondary torsions, were defined in terms of the squaring operations of Steenrod [Ann. of Math. (2) 48, 290–320 (1947); these Rev. 9, 154] and together with the classical numerical invariants (the Betti numbers and torsion coefficients) determine the homotopy type of P. In the present paper the authors define similar numerical invariants for any finite polyhedron K. They are also defined in terms of the squaring operations and are called the block invariants of K. If K^n denotes the n-dimensional skeleton of K and K_n denotes the complex obtained by shrinking K^{n-1} to a point, the authors show how the block invariants of K are related to those of K^n and K_n. Using these results and the results of Chang, it is shown that if $n>2$ the block invariants and the classical numerical invariants determine $\pi^{n+1}(K)$, if $\dim K \leqq n+2$, and $\pi_{n+1}(K)$ if $\pi_r(K)=0$ for $r \leqq n-1$. *E. H. Spanier* (Chicago, Ill.).

Postnikov, M. M. Determination of the homology groups of a space by means of the homotopy invariants. Doklady Akad. Nauk SSSR (N.S.) 76, 359–362 (1951). (Russian)

The object of this note is to give a set of invariants which, together with the homotopy groups of an (arcwise-connected) space X, determine its singular homology groups.

The author defines a sequence of abstract cell-complexes, $K_1, K_2, \cdots, K_n, \cdots$, with $K_1 = K(\pi_1(X))$, in the sense of Eilenberg-MacLane, and a sequence of "characteristic" co-

cycles $k_1, k_2, \cdots, k_n, \cdots, k_n \in Z^{n+2}(K_n, \pi_{n+1}(X))$, such that K_{n+1} is, in a complicated but definite sense, the augmentation of K_n by k_n (in fact, $K_n^n = K_{n+1}^n$, $K_n^{n+1} \subset K_{n+1}^{n+1}$, K^r being the r-section of the cell-complex K). The author also defines "normal" singular simplexes, generalizations of the singular simplexes of the Eilenberg-MacLane complex $S_1(X)$ (the latter consisting of maps $T: \sigma \to X$ sending all the vertices of σ into the base point of X), and asserts (i) that the natural embedding of the normal singular complex $N(X)$ in the singular complex $S(X)$ is an equivalence, so that these complexes have isomorphic homology groups, and (ii) that the limit complex of the sequence $K_1, K_2, \cdots, K_n, \cdots$ is, in a precise sense, isomorphic to $N(X)$. Thus the homotopy groups of X, together with the cocycles $k_1, k_2, \cdots, k_n, \cdots$, determine the singular homology groups of X. Proofs of these assertions are not given. *P. J. Hilton.*

Postnikov, M. M. On the homotopy type of polyhedra. Doklady Akad. Nauk SSSR (N.S.) **76**, 789–791 (1951). (Russian)

The author uses the results of a previous paper [cf. the preceding review; this paper will be referred to henceforth as P1] to obtain complete systems of invariants of homotopy type and n-type of a CW-complex, which he calls a polyhedron [for definitions of CW-complex, n-type, see J. H. C. Whitehead, Bull. Amer. Math. Soc. **55**, 213–245 (1949); these Rev. **11**, 48].

The author introduced in P1 the natural system $K_1, K_2, \cdots, K_n, \cdots; k_1, k_2, \cdots, k_n, \cdots$, of a space X. Such a system may be defined purely algebraically in terms of a sequence of groups $G_1, G_2, \cdots, G_n, \cdots$, in which G_i, $i \geqq 2$, is an additive abelian group admitting G_1 as a group of operators. In this paper the author proves the following theorems. (1) Every (arcwise-connected) space dominated by a polyhedron is of the same homotopy type as some polyhedron. (2) Every system $(K_1, K_2, \cdots, K_n, \cdots; k_1, k_2, \cdots, k_n, \cdots)$ is, in a precise sense, n-isomorphic $(1 \leqq n \leqq \infty)$ to the natural system of some n-dimensional polyhedron. (3) The n-types $(1 \leqq n \leqq \infty)$ of two polyhedra coincide if and only if the natural systems of these polyhedra are n-isomorphic. (1) is an immediate consequence of a result due to J. B. Giever [Ann. of Math. (2) **51**, 178–191 (1950); these Rev. **11**, 379]. (2) and (3) are deduced from the results of P1, and constitute generalizations of the results of S. MacLane and J. H. C. Whitehead on 3-type [Proc. Nat. Acad. Sci. U. S. A. **36**, 41–48 (1950); these Rev. **11**, 450]. *P. J. Hilton.*

Postnikov, M. M. On the classification of continuous mappings. Doklady Akad. Nauk SSSR (N.S.) **79**, 573–576 (1951). (Russian)

In this note the author extends his application of the "natural system" of a space [cf. two previous reviews; the former will be referred to as P1] to solve the homotopy classification problem for maps of a polyhedron P into an arcwise-connected space X. In this note, "polyhedron" is to be taken as meaning "locally-finite simplicial polyhedron", although the author states that his results may easily be extended to CW-complexes.

Given a natural system of groups $(G_1, G_2, \cdots, G_n, \cdots)$ and cohomology classes $(k_1, k_2, \cdots, k_n, \cdots)$, the author defines a "cycloid" over the system (or, as we would say, a cocycloid) as a system $(C^1, \cdots, C^n, \cdots)$ where C^n is a n-cochain of P over G_n. Here C^1 is actually a cocycle and the condition that a "chainoid" be a cycloid involves the k_r. A very complicated condition is then given to describe when two cycloids shall be called "homologous". A "normal" mapping, f, of P into X is one such that its restriction to a n-simplex of P determines a normal singular n-simplex of X (cf. P1). Any mapping of P into X is homotopic to a normal mapping, and it is shown that any normal mapping of P into X determines a cycloid of P over the natural system of X. Then the author asserts (i) that any cycloid of P over the natural system of X is that determined by some normal map of P into X, and (ii) that two normal maps of P into X are homotopic if and only if the cycloids they determine are homologous. It is also stated that if $\dim P = n+1$, and $\pi_r(X) = 0$, $1 \leqq r < n$, then the factor k_n may be described explicitly. The paper does not contain proofs.

P. J. Hilton (Manchester).

Torres Díaz, Guillermo. On orientable surfaces extendible into knots. Bol. Soc. Mat. Mexicana **8**, 1–14 (1951). (Spanish)

To any regular projection k_π of a connected polygon k (into a plane π) there is associated a (non-singular) orientable surface F whose boundary is k. This surface, which is unique up to homeomorphisms of 3-space on itself that leave k fixed, is constructed from k_π by an algorithm described by Seifert [Math. Ann. **110**, 571–592 (1934)]. The question: Can one obtain by this algorithm every type of surface in 3-space? is here answered in the negative. On the one hand the author proves, by an ingenious argument involving a representation of the groups of a whole class of knots by rigid motions of the hyperbolic plane, that any surface of genus one whose boundary curve is unknotted must be transformable, by a homeomorphism of 3-space on itself, into a subset of the unknotted torus $(r-2)^2 + z^2 = 1$. On the other hand he constructs a surface of genus one whose boundary is unknotted and on which one can draw a simple closed curve that is not a torus knot.

R. H. Fox (Utrecht).

GEOMETRY

Thébault, Victor. On Feuerbach's theorem. Amer. Math. Monthly **58**, 620–622 (1951).

Tallqvist, Hj. Regular star-shaped polygons. Soc. Sci. Fenn. Comment. Phys.-Math. **14**, no. 15, 8 pp. (1950). (Swedish)

These are the simple but non-convex $2n$-gons formed by the peripheral portions of the sides of the star polygons $\{n/d\}$ [Coxeter, Regular Polytopes, Pitman, New York, 1949, p. 93; these Rev. **10**, 261]. The author calls such a polygon $P_{n,i}$, where $i = d-1$, and observes that it has alternate angles $(1-2d/n)\pi$ and $(1+2i/n)\pi$. He tabulates these angles to the nearest second, and the chief metrical properties to five decimal places, for the cases $n \leqq 16$.

H. S. M. Coxeter (Toronto, Ont.).

***Stark, Marceli. Geometria analityczna ze szczególnym uwzględnieniem podręcznika Eustachego Żylińskiego. [Analytic Geometry with Particular Regard to the Textbook of Eustachy Żyliński].** Monografie Matematyczne. Tom XXVI. Polskie Towarzystwo Matematyczne., Warszawa-Wrocław, 1951. viii+629 pp.

This textbook deals with metric, affine and projective geometry of linear and quadratic varieties in the plane as

well as in the three-space. The author does not confine himself only to real plane (real space) but considers also the complex plane (complex space). Besides items usually dealt with in textbooks of elementary analytic geometry the reader finds here the introduction to synthetic projective geometry, to the theory of matrices (and determinants) with the usual applications and to the (three-dimensional) elementary vector calculus. *V. Hlavatý.*

Morduhaĭ-Boltovskiĭ, D. D. Parallelism and perpendicularity of straight lines, planes and hyperplanes in three-dimensional and four-dimensional Lobačevskiĭ spaces. Uspehi Matem. Nauk (N.S.) **6**, no. 4(44), 176–183 (1951). (Russian)

Remarks on the subject indicated in the title.

H. Busemann (Auckland).

Pimiä, Lauri. Zur Geometrie der Kreise oder Punktepaare in komplexen Raum. Soc. Sci. Fenn. Comment. Phys.-Math. **14**, no. 17, 21 pp. (1949).

In a previous paper [same Comment. **14**, no. 5 (1948); these Rev. **10**, 394] the author dealt with a mapping of the points of the three-dimensional complex space on the system of circles of the three-dimensional real space. This mapping is now employed to study the complex circles by means of bicomplex, pentaspherical coordinates. A pair of oriented real circles is introduced as the real representative of a complex circle and is used in discussing properties of complex circles and also in investigating orthogonal linear transformations. *E. Lukacs* (Washington, D. C.).

Pimiä, Lauri. Über die nichteuklidische Liniengeometrie im komplexen Raum. Soc. Sci. Fenn. Comment. Phys.-Math. **14**, no. 18, 7 pp. (1949).

Using the methods of his earlier papers [see preceding review] the author studies the geometry of lines in a non-euclidean (elliptic or hyperbolic) complex space.

E. Lukacs (Washington, D. C.).

Bachmann, Friedrich, und Klingenberg, Wilhelm. Über Seiteneinteilungen in affinen und euklidischen Ebenen. Math. Ann. **123**, 288–301 (1951).

Let $\mathfrak{p}$ be an affine plane over a skew field. An ordered point triple consists of three collinear points P_0, P_1, P_2 $[P_1 \neq P_0 \neq P_2]$. A betweenness function $P_0(P_1, P_2) = \pm 1$ is defined for every ordered triple. It is invariant under parallel projection and satisfies a transitive law

$$P_0(P_1, P_2) \cdot P_0(P_2, P_3) = P_0(P_1, P_3).$$

An order function $g(P_1, P_2) = \pm 1$ is defined for every straight line g and every pair of points P_1, P_2 outside of g. It satisfies the corresponding transitive law. Furthermore $g(P_1, P_2) = 1$ if $P_1P_2 \| g$; and $g(P_1, P_2) = g'(P_1, P_2)$ if $P_0 = g \cap g'$, P_1, P_2 is an ordered point triple. There exists a one-one correspondence between the betweenness functions and the order functions in $\mathfrak{p}$.

A semiordering of a group $\mathfrak{G}$ is defined by means of a quadratic character $\chi(a)$ [thus $\chi(a) = \pm 1$ is defined for every $a \subset \mathfrak{G}$, and $\chi(ab) = \chi(a) \cdot \chi(b)$]. Let $\mathfrak{Q}$ be the invariant subgroup of all those elements of $\mathfrak{G}$ that can be expressed as products of squares. Then $\mathfrak{G}/\mathfrak{Q}$ is abelian, each element being of order two. The group of the quadratic characters of $\mathfrak{G}$ is isomorphic to the well known character group of $\mathfrak{G}/\mathfrak{Q}$. Hence, $\mathfrak{G}$ possesses non-trivial semiorderings [$\chi(a) = -1$ for some $a \subset \mathfrak{G}$] if and only if $\mathfrak{Q} \neq \mathfrak{G}$.

Each betweenness relation in $\mathfrak{p}$ induces a semiordering of the multiplicative group of the skew-field of $\mathfrak{p}$ and vice-versa. The trivial semiordering corresponds to the trivial betweenness relation [$P_0(P_1, P_2) \equiv 1$]. These concepts and results are applied to $\mathfrak{p}$'s in which Pascal's theorem holds true and to certain euclidean planes. The following theorem about the latter may be quoted: There exists a betweenness function $P_0(P_1, P_2)$ which is equal to -1 if P_0 is the intersection of the hypotenuse P_1P_2 of a right triangle with its altitude.

P. Scherk (Saskatoon, Sask.).

Hoffman, A. J. A note on cross ratio. Amer. Math. Monthly **58**, 613–614 (1951).

Given the projective line over the field F. It is shown very neatly that a one-to-one mapping of the line onto itself which preserves cross ratios with a specified value k will be the product of a projective transformation and an automorphism of F. *Marshall Hall* (Washington, D. C.).

Skornyakov, L. A. Natural domains of Veblen-Wedderburn projective planes. Amer. Mat. Soc. Translation no. **58**, 37 pp. (1951).

Translated from Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 447–472 (1949); these Rev. **11**, 198.

Hoffman, Alan J. On the foundations of inversion geometry. Trans. Amer. Math. Soc. **71**, 218–242 (1951).

Über einem geordneten Körper K, in dem jede nicht negative Zahl ein Quadrat ist, kann man eine dreidimensionale Inversionsgeometrie Π definieren mit den Raumelementen Punkt, Kreis, Kugel und gesamter Inversionsraum. Es wird ein System von Axiomen aufgestellt, aus dem sich folgende Aussagen ableiten lassen. 1) Zu einem Punkt p existiert eine affine Geometrie, deren Pseudopunkte, Pseudogeraden, Pseudoebenen und Pseudo-R_3 gebildet werden resp. von den Punkten von Π mit Ausnahme von p, den Kreisen durch p, den Kugeln durch p und dem Inversionsraum. 2) Der dieser affinen Geometrie zugrunde liegende Körper ist K. 3) Die affine Geometrie kann zu einer euklidischen Geometrie gemacht werden, so, dass die Pseudokreise und Pseudokugeln der euklidischen Geometrie die Kreise und Kugeln von Π sind, die nicht durch p gehen. Die Verallgemeinerung der Untersuchungen auf höhere Dimensionen als 3 bietet keine Schwierigkeiten; die Dimension 2 wird in einer späteren Arbeit behandelt. Verf. geht aus von der verbandstheoretischen Begründung der Geometrie nach Birkhoff und Menger [siehe z.B. Birkhoff, Lattice Theory, rev. ed., Amer. Math. Soc. Colloq. Publ., v. 25, 1948; diese Rev. **10**, 673] und gelangt so stufenweise zu einer Algebraisierung seiner Inversionsgeometrie. Die erste Gruppe von vier Axiomen umfasst im wesentlichen Inzidenzaxiome; ein weiteres Axiom bezieht sich auf die Existenz von Fixpunkten der Inversion. Das letzte Axiom übernimmt die Rolle eines Schnittpunktsatzes und kann ersetzt werden durch die Forderung (Satz von Miquel): Wenn sich aus vier Paaren verschiedener Punkte auf fünf Weisen zwei Paare so auswählen lassen, dass diese vier Punkte auf einem Kreis liegen, so ist das auch bei der sechsten Auswahl der Fall.

R. Moufang (Frankfurt a.M.).

Stettler, R. Über endliche Geometrien. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **72**, 45 pp. (1950).

Using Hilbert's calculus of segments, the author shows that the finite affine geometry $EG(2, p^s)$ is characterized by the following axioms:

Any two points determine a unique line. Every line contains at least two points. For any line there is a point not on it, and through any such point there is a unique parallel (i.e., a line not meeting the given line). The number of points is finite. If two triangles are in perspective from a point while two pairs of corresponding sides are parallel, the remaining sides are parallel.

He considers the consequence of weakening the last axiom to the "little axiom of Desargues," where the joins of corresponding vertices of the two triangles are not concurrent but parallel. He shows that the consequent algebra is a Veblen-Wedderburn system [O. Veblen and J. H. M. Wedderburn, Trans. Amer. Math. Soc. 8, 379–388 (1907)]. He deduces the closure of Thomsen's hexagon, and a special case of the theorem of the complete quadrangle. However, most of these results were anticipated by Marshall Hall [Trans. Amer. Math. Soc. 54, 229–277 (1943); these Rev. 5, 72] and L. A. Skornyakov [Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 447–472 (1949); these Rev. 11, 198; for a translation see the second preceding review].

H. S. M. Coxeter (Toronto, Ont.).

Ellis, David. Notes on abstract distance geometry. II. Implications of basality in generalized semimetric spaces. Monatsh. Math. 55, 185–187 (1951).

S and G are abstract sets and $d(x, y)$ (generalized distance function) a single-valued mapping of $S \times S$ into G. Assumptions: $d(x, y) = d(y, x)$, $d(x, x) = d(y, y) \neq d(x, y)$ for each pair of distinct elements of S. Basality: Corresponding to any p in S and any d in G, there exists exactly one element q of S with $d(p, q) = d$. Assertions: 1) For any two points x, y of S there exists a d-preserving mapping of S onto S (motion) carrying x into y (simple transitivity). 2) Any d-preserving mapping of a subset of S into S (congruence) may be extended to a motion. 3) S is not congruent with any proper subset of itself. 4) There is no proper subset S^* of S with the same distance set $D(S) = d(S \times S)$. 5) $D(S) = G$.

C. Y. Pauc (Cape Town).

Ellis, David. Autometrized Boolean algebras. I. Fundamental distance-theoretic properties of B. Canadian J. Math. 3, 87–93 (1951).

The author is interested in furthering the study of metric spaces where the distance values need not be numbers. An autometrized Boolean algebra is defined to be a Boolean algebra where the distance of two elements is what is traditionally called their symmetric difference. The author discusses the analogues for this distance of the very elementary properties of the usual metric spaces. Metric betweenness, congruences and motions are also mentioned.

L. Nachbin (Rio de Janeiro).

Ellis, David. Autometrized Boolean algebras. II. The group of motions of B. Canadian J. Math. 3, 145–147 (1951).

[Cf. the preceding review.] In this article the author remarks that the group of motions $M(B)$ of a Boolean algebra B and the group of authomorphisms of B are disjoint subgroups of the group of all complementation-preserving permutations of the set B; and that $M(B)$ and the additive group of the Boolean ring of B are isomorphic.

L. Nachbin (Rio de Janeiro).

Convex Domains, Extremal Problems, Integral Geometry

Hanner, Olof, and Rådström, Hans. A generalization of a theorem of Fenchel. Proc. Amer. Math. Soc. 2, 589–593 (1951).

The points and sets considered are to lie in an n-dimensional real vector space. Say that a set k-simplicially covers a point if that point lies in a possibly degenerate k-simplex with vertices in the set. Call a set k-simplicial if it k-simplicially covers every point of its convex hull. Thus, as is known, every set is n-simplicial. What sets are $(n-1)$-simplicial? Fenchel [Math. Ann. 101, 238–252 (1929)] showed that any compact connected set is; and Bunt [Bijdrage tot der theorie der puntverzamelingen, Thesis, Groningen, 1934, ch. 1] showed that any set having at most n components is. These results are blended as follows. Call a set convexly connected if it cannot be separated by a plane; and define convex components of a set. Then, any compact set having at most n convex components is $(n-1)$-simplicial.

W. Gustin (Bloomington, Ind.).

Garnier, René. Sur un théorème de Schwarz. Comment. Math. Helv. 25, 140–172 (1951).

The result stated without proof by H. A. Schwarz [Gesammelte mathematische Abhandlungen, Band I, Springer, Berlin, 1890, p. 111], that a skew quadrilateral cannot bound more than one minimal surface of the type of the circular disc and having no singularities in its interior, follows as a corollary from a theorem of T. Radó [Proc. Nat. Acad. Sci. U. S. A. 16, 242–248 (1930)] concerning minimal surfaces spanning contours having convex central or parallel projections on a plane. The author now gives the details of a previously announced alternative proof [C. R. Acad. Sci. Paris 217, 60–62, 420–421 (1943); these Rev. 6, 21, 187], by means of the theory of differential equations, of the above result concerning skew quadrilaterals.

E. F. Beckenbach (Princeton, N. J.).

Hadwiger, H. Zum Problem der Zerlegungsgleichheit der Polyeder. Arch. Math. 2 (1949–1950), 441–444 (1951).

Let a_ν, b_μ be the edges of two polyhedra A, B in E_3 respectively. Denote by $\alpha_\nu(\beta_\mu)$ the angle between the exterior normals of the faces of $A(B)$ having $a_\nu(b_\mu)$ as side. If $\omega_0 = \pi, \omega_1, \cdots, \omega_n$ are suitably chosen rationally independent numbers such that α_ν and β_μ can be represented uniquely in the form $\alpha_\nu = \sum_{\lambda=0}^{n} p_{\lambda\nu}\omega_\lambda$, $\beta_\mu = \sum_{\lambda=0}^{n} q_{\lambda\mu}\omega_\lambda$, where the $p_{\lambda\nu}$ and $q_{\lambda\mu}$ are rational, then the relations

$$\sum_\nu p_{\lambda\nu} a_\nu = \sum_\mu q_{\lambda\mu} b_\mu, \quad \lambda = 1, \cdots, n,$$

are necessary for A and B to be decomposable into congruent pairwise congruent polyhedra.

H. Busemann.

Valentine, F. A. A characteristic property of the circle in the Minkowski plane. Amer. Math. Monthly 58, 484–487 (1951).

A circular disc may be characterized among compact sets in a Minkowski plane by the following property: every two of its points are endpoints of some semicircular arc lying in the set.

W. Gustin (Bloomington, Ind.).

***Santaló, L. A. Integral geometry in general spaces.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 483–489. Amer. Math. Soc., Providence, R. I., 1952.

A survey, by means of typical examples, of different aspects of integral geometry, including convex bodies in

unimodular affine space, integral geometry in spaces of constant curvature, Hermitian spaces, and Riemannian spaces, and its application to the definition of the p-dimensional measure in n-dimensional Euclidean space. *S. Chern.*

Algebraic Geometry

Huff, Gerald B. On the existence of plane curves with prescribed singularities. Bull. Amer. Math. Soc. **57**, 411–419 (1951).

Expository lecture. *J. A. Todd* (London).

Lorent, H. Correspondances (1, 4) entre points du plan ou de l'espace. Bull. Soc. Roy. Sci. Liège **20**, 379–389 (1951).

Two circles, one with center at $A(x', y')$ and given radius, the other through the origin and with center $(x', -y')$, determine a (1, 4) correspondence between the points A and the points of intersection $B(x, y)$ of the pairs of circles. Loci of coincidences and the images of lines parallel to the coordinate axes are obtained. The author extends this method to conics in which case all four points B may be real. The space generalization of the above is not a (1, 4) point correspondence. It is defined by two spheres, one with center at $A(x', y', z')$ and given radius and the other through the origin and with center at $(x', y', -z')$, and "to the point A corresponds the circle common to the two spheres". The surfaces generated by this common circle are obtained when A describes lines parallel to the axes and circles parallel to the coordinate planes. *T. R. Hollcroft* (Aurora, N. Y.).

Lorent, H. Transformation agnésienne de lignes et de surfaces. Bull. Soc. Roy. Sci. Liège **20**, 120–132 (1951).

Meyler, Dorothy S. A point representation of a system of rational normal curves of order n through $n+1$ fixed points. Quart. J. Math., Oxford Ser. (2) **1**, 33–40 (1950).

The author investigates rational normal n-ics passing through $n+1$ fixed points and having a given $(n-1)$ secant secundum or a given chord by using a point representation in which the n-ics through $n+1$ fixed points having a given $(n-1)$ secant secundum are represented by the points of a prime, and those having a given line as chords by the points of a plane. *M. Piazzolla Beloch* (Ferrara).

Arvesen, Ole Peder. Sur certaines surfaces algébriques, parmi lesquelles la surface de Steiner constitue le cas le plus simple. Norske Vid. Selsk. Forh., Trondheim **22**, no. 42, 198–201 (1950).

L'A. considère les surfaces polaires, du plan de l'infini, par rapport à la surface réductible de n^{me} classe, se composant de n points ($n \geqq 4$), et montre que pour $n=4$ points, non tous situés dans le même plan, on obtient la surface de Steiner. *M. Piazzolla Beloch* (Ferrara).

Turri, Tullio. I sistemi lineari di cubiche a modulo costante. Rend. Sem. Fac. Sci. Univ. Cagliari **20** (1950), 1–29 (1951).

Les cubiques ayant un module donné forment une variété W à 8 dimensions de l'espace E_9 dont les points représentent les cubiques du plan; tout système linéaire de cubiques à module constant se représente par une variété linéaire contenue dans W, les homographies du plan qui conservent ce système induisent dans E_9 des transformations homographiques conservant ces variétés linéaires. L'auteur admet que par continuité il existe des homographies de E_9 qui là laissent une de ces variétés linéaires S_d fixe, ces homographies forment un groupe à paramétres. L'étude des groupes de ce type contenus dans celui des homographies planes et de leurs sous-groupes commutatifs, montre qu'ils doivent contenir des homographies cycliques conservant projectivement une cubique; avec leurs puissances, ceci les ramène aux homologies harmoniques, aux homologies cycliques du 3° ordre, aux homographies cycliques du 3° ordre.

Formant alors et discutant les équations des cubiques conservées par ces homographies, on obtient deux faisceaux de cubiques à modules généraux conservés par homologie harmonique, deux réseaux de cubiques harmoniques conservées par homologie harmonique, un réseau de cubiques équianharmoniques conservées par homographie cyclique du 3° ordre, un système ∞^4 de cubiques équianharmoniques conservées par homologie cyclique du 3° ordre, enfin un faisceau de cubiques équianharmoniques conservées par homologie harmonique *B. d'Orgeval* (Alger).

Turri, Tullio. Inesistenza di trasformazioni piane cicliche con sei punti fondamentali. Rend. Sem. Fac. Sci. Univ. Cagliari **20** (1950), 30–35 (1951).

S. Kantor [Acta Math. **19**, 115–193 (1895)] pensait établir l'existence de transformations planes cycliques avec six points fondamentaux, à partir de l'existence de transformations cycliques de l'espace conservant une surface cubique générale. Reprenant ces homographies spatiales, l'auteur montre qu'il n'y a à étudier que celle dont la matrice canonique est $(1, 1, 1, \epsilon)$, $\epsilon^3 = 1$, homologie centrale, de centre extérieur à la cubique. Si l'on prend le plan d'homologie pour plan de la représentation plane, si la transformation spatiale est image d'une du plan telle celle de Kantor, les droites de la cubique donnant les points-base de la représentation devraient se conserver, ce qui est impossible.

B. d'Orgeval (Alger).

Turri, Tullio. Sulla costruzione delle trasformazioni involutorie del piano. Rend. Sem. Fac. Sci. Univ. Cagliari **20** (1950), 36–42 (1951).

En remarquant que dans une transformation de Jonquières, $D \to C_n(O^{n-1}, A_1, \cdots; A_{2n-2})$, les courbes d'ordre s passant $s-2$ fois en O et simplement aux points A se conservent, l'auteur montre que les couples de l'involution sont à l'intersection d'une droite issue de O et d'une courbe d'un faisceau de courbes rationnelles unies. Pour les involutions de Geiser et de Bertini, ces couples sont à l'intersection respectivement de cubiques de deux faisceaux de courbes unies et de sextiques appartenant à des faisceaux de courbes unies. *B. d'Orgeval* (Alger).

Andreotti, A. Sopra le involuzioni appartenenti ad una varietà di Picard. Pont. Acad. Sci. Acta **14** (1950), 107–115 (1951). (Italian. Latin summary)

The author proves the two following theorems, which are generalisations of classical results for hyperelliptic surfaces. (1) A variety W_p of superficial irregularity p, which is the image of an involution on a Picard variety V_p, is itself a Picard variety, and the involution on V_p is generated by a group of birational transformations of the second kind. (2) Every involution on a Picard variety V_p possessing at most ∞^{p-2} coincidences, whose image has at least one plurigenera equal to one, and which is not compounded of an abelian involution of rank one, can be generated by a group of birational transformations of V_p into itself. An example is given

to show that the last condition in the statement of (2) is essential. The proofs are transcendental. *J. A. Todd.*

✓*Dubreil, Paul. **La fonction caractéristique de Hilbert.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 109–114. Centre National de la Recherche Scientifique, Paris, 1950.

Various results are announced concerning Hilbert's function, and some unsolved problems are pointed out. The theorems are related to certain results of Sperner [Abh. Math. Sem. Univ. Hamburg **7**, 149–163 (1929)], but the precise statements are somewhat too complicated to be reproduced here. They are also related to the theorem of Halphen giving the maximum genus for a space curve of given degree. In these terms a condition is also announced for a given 1-dimensional homogeneous ideal in a polynomial ring $K[X_0, X_1, X_2, X_3]$ to represent a plane curve. *I. S. Cohen* (Cambridge, Mass.).

Franchetta, Alfredo. Forme algebriche sviluppabili e relative hessiane. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 386–389 (1951).

Let F^d be a hypersurface in the complex projective $(d+1)$-space, and let H be the Hessian divisor of F, i.e. the divisor having for equation the determinant of the second derivatives of the equation of F. It is well known that, for F to be a component of H, it is necessary and sufficient that F be a developable hypersurface (and therefore F contains a system S of ∞^{d-1} straight lines). The case where F is a multiple component of H is studied here. A necessary and sufficient condition for that is that the residual intersection of F with a generic plane containing the generic line L of S be a curve intersecting L in only one point of multiplicity $n-1$ (n being the degree of F). When $d=2$ the only possible developable surface F has a skew cubic as an edge of regression. For $d\geqq 3$, the study of the foci of the system S leads to the conclusion that F must be of degree $(d-1)^2$. A proof of this is given in the cases $d=3$, $d=4$. The general case, and the existence of hypersurfaces F multiply contained in their Hessian divisor, will be studied in an other paper. A discussion of hypersurfaces F whose Hessian divisor is indeterminate is given here for $d=3$ and 4. The reviewer notices that these results, which pertain to differential geometry, are not to be applied to algebraic varieties over fields of characteristic $p\neq 0$. *P. Samuel* (Clermont-Ferrand).

Segre, Beniamino. Sulla perfezione delle coincidenze isolate. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 335–336 (1951).

If T is an analytical ∞^n correspondence of an analytical V_n into itself, and there is an isolated coincidence in a simple point of V_n, it is proved that such an isolated coincidence is always perfect [the reference in footnote (3) is to these Rev. **12**, 529]. *D. Pedoe* (London).

Segre, Beniamino. On the perfection of isolated coincidences. Revista Mat. Hisp.-Amer. (4) **11**, 129–131 (1951). (Spanish)

A translation of the paper reviewed above.

Akizuki, Yasuo. Theorems of Bertini on linear systems. J. Math. Soc. Japan **3**, 170–180 (1951).

The fundamental theorem of Bertini on the variable singular points of a linear system of varieties was proved in a revised form by Zariski [Trans. Amer. Math. Soc. **56**, 130–140 (1944); these Rev. **6**, 186] who pointed out that the original formulation of the theorem fails to hold in general for varieties over fields of characteristic $p\neq 0$. If $\lambda_0 f_0(x)+\lambda_1 f_1(x)+\cdots+\lambda_m f_m(x)=0$ defines a linear system of U_{r-1}'s on a variety U_r with general point (x), the author defines a set of "critical" subvarieties Φ of U_r with the aid of the derivations of $k(x)$ over k. If (x') is a general point of Φ, if $\bar{k}$ is the algebraic closure of the ground field k, and if when $f_s(x')\neq 0$, $y_i'=f_i(x')/f_s(x')$, then a sufficient condition for Bertini's theorem to hold in its original form is that $\bar{k}(x')$ be separably generated over $\bar{k}(y')$ for every critical variety Φ. The paper also includes another proof of Matsusaka's extension to modular fields [Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **26**, 51–62 (1950); these Rev. **12**, 853] of Zariski's proof of Bertini's theorem on reducible systems, and a new proof of this theorem in the classical case. *H. J. Muhly* (Iowa City, Ia.).

Nakai, Yosikazu. Note on the intersection of an algebraic variety with the generic hyperplane. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **26**, 185–187 (1951).

Let V be an algebraic variety, W the intersection of V with the general hyperplane of the ambient space. It is shown that any point of W which is simple for V is also simple for W. For characteristic 0, this is deduced from the second theorem of Bertini: but no such elaborate theorem is needed for a section by a hyperplane, and indeed the simple computation of the note suffices also for arbitrary characteristic. Still, the value of a theorem depends on the definitions. In the present note, the classical Jacobian matrix condition is taken as defining simple point: in the case of characteristic p, however, a paper of Zariski [Trans. Amer. Math. Soc. **62**, 1–52 (1947); these Rev. **9**, 99] should be consulted, where the limited value of this classical condition is established. *A. Seidenberg* (Berkeley, Calif.).

Matsusaka, Teruhisa. Specialization of cycles on a projective model. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **26**, 167–173 (1951).

To every cycle $X=X_1-X_2$ (X_1, X_2 positive) in the projective space is associated its "Chow-point" $(x)=(x_1, x_2)$, where (x_i) $(i=1, 2)$ is the point (of a suitable projective space) having as homogeneous coordinates the coefficients of the associated form of X_i [cf. Chow and van der Waerden, Math. Ann. **113**, 692–704 (1937)]. A cycle X' is said to be a specialization of X over a field k if its Chow-point (x') is a specialization of (x) over k. A lemma shows that, if a cycle X' (on a variety V) is a nongeneric specialization of X, then there exist a curve C, a generic point M of C, a simple point M' of C, and a cycle Y on $V\times C$ such that $X\times M=(V\times M)\cdot Y$ and $X'\times M'=(V\times M')\cdot Y$. The converse (with a variety instead of a curve) is also proved. From this it is easy to deduce that the notion of specialization of cycles fulfills the conditions imposed by A. Weil [Foundations of Algebraic Geometry, Amer. Math. Soc. Colloq. Publ., v. 29; New York, 1946; these Rev. **9**, 303]: it is compatible with the operations of addition, product, intersection product (when defined) and birational transformation (where biregular). These facts are proved only in the case of cycles on a nonsingular variety, a restriction one could get rid of. The statement that a cycle is rational over k when its Chow-point is (lemma 1, p. 168) is not true in characteristic $p\neq 0$ if one follows A. Weil's definition of rational cycles; but it does not seem to the reviewer that this impairs the validity of the compatibility theorems. *P. Samuel.*

Schmid, Josef. Ein idealtheoretischer Beweis des Kriteriums von Plücker-Clebsch. Monatsh. Math. **55**, 233–241 (1951).

The proof of the principle, in its formulation due to Severi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **17**, 3–10 (1933); see also the proofs of van der Waerden, Math. Ann. **110**, 134–160 (1934), and Terracini, same Rend. (6) **21**, 651–655 (1935)] is based on the well known fact [see van der Waerden, Math. Ann. **108**, 113–125 (1933)] that all isolated components of an ideal $(f_1, \cdots, f_r)$ of $K[x_1, \cdots, x_n]$, where the x_i are independent indeterminates over K, have a dimension $\geqq n-r$. A proof of this proposition is also given. *P. Abellanas* (Madrid).

Igusa, Jun-ichi. Algebraic correspondences between algebraic varieties. J. Math. Soc. Japan **3**, 215–219 (1951).

Given two complete nonsingular varieties U and V, a divisor X on $U\times V$ is called a correspondence between U and V. The correspondence X is said to be with valence zero if it is linearly equivalent to a degenerate one (i.e. of the form $(C\times V)+(U\times D)$). The factor module of the module of all correspondences by the submodule of correspondences with valence zero is called the correspondence module of U and V, and is denoted $C(U, V)$. When the universal domain is the field of complex numbers, it is proved that $C(U, V)$ is isomorphic with the module $H(A, B)$ of homomorphisms of the abelian variety A into the abelian variety B, A and B being any abelian varieties isogeneous (i.e. locally isomorphic) to the Picard varieties of U and V. This result is to be proved algebraically in a forthcoming paper of the author, by taking for A the Albanese variety of U (i.e. the "universal" abelian variety for mappings of U into abelian varieties) and for B the Picard variety of V. In the present paper a direct proof is given. Supposing that U and V are projective varieties, and using a Kählerian metric on $U\times V$, the theory of harmonic integrals permits one to associate to every correspondence X on $U\times V$ an integral matrix $S(X)$ such that $\Omega_U \cdot S(X) \cdot {}^t\Omega_V = 0$ (Ω_U being the period matrix of the Picard differential of the first kind on U over a base of the one-dimensional cycles). The kernel of $X \to S(X)$ is the module of correspondences with valence zero, and classical results on abelian varieties lead to the conclusion. *P. Samuel* (Clermont-Ferrand).

Hodge, W. V. D. The characteristic classes on algebraic varieties. Proc. London Math. Soc. (3) **1**, 138–151 (1951).

In studying relations between the curvature of an Hermitian metric on a complex analytic manifold with the complex structure, the reviewer [Ann. of Math. (2) **47**, 85–121 (1946); these Rev. **7**, 470] introduced certain cohomology classes, called the characteristic classes of the manifold. The purpose of this paper is to identify them, for the case of non-singular algebraic varieties over the complex field, with the canonical systems of Eger and Todd. The proof is carried out in detail for the cases: 1) Non-singular algebraic varieties of dimension m, such that, for $r=1, \cdots, m$, there are at least r linearly independent Picard integrals $\int \sum_a P_{ia} dz_a$, $i=1, \cdots, r$, having the property that the locus of points at which the matrix $\|P_{ia}\|$ is of rank $r-1$ is of complex dimension $r-1$, with simple components; 2) varieties which are complete intersections of non-singular hypersurfaces. Proof for the general case is only indicated and not claimed to be complete, owing partly to the sketchy nature of the theory of canonical systems. An adjunction formula for characteristic cycles of sub-varieties is proved, which should be of independent interest. *S. Chern.*

Nakano, Shigeo. On invariant differential forms on group varieties. J. Math. Soc. Japan **2**, 216–227 (1951).

In the first part of the paper, the author extends to higher dimensional algebraic varieties V the notion of differential form, which had been introduced by A. Weil in the case of curves [Sur les courbes algébriques et les variétés qui s'en déduisent, Hermann, Paris, 1948; these Rev. **10**, 262]. Roughly speaking, a differential form on V is a coset of the group of functions on $V\times V$ which vanish along the diagonal D modulo the subgroup of those which vanish "of the second order" along D. The differential forms form a module over the field R of rational functions on the variety, which is dual to the module of derivations of R. The notion of coboundary (or differential) may be extended to differential forms on a variety.

Let now G be a group variety. To every $a \in G$ there corresponds a left translation T_a on G, which induces a transformation δT_a of the module of differential forms into itself. A differential form is said to be left-invariant if it is transformed into itself by every operation δT_a. The author proves that, if G is of dimension n, then there are n linearly independent left-invariant differential forms on G, say $\omega_1, \cdots, \omega_n$, which form a base of the module of differential forms. These may be obtained as follows: selecting a coordinate system composed of functions which are 0 at the unit element of G, the coordinates of $x^{-1}y$, considered as functions on $G\times G$ vanish on the diagonal and yield left-invariant differential forms, of which n are linearly independent.

The characteristic of the basic field being assumed to be $\neq 2$, we may write $d\omega_i = \frac{1}{2}\sum_{jk} c_{ijk}\omega_j\omega_k$, the c_{ijk}'s being constants. These constants are constants of structure of a Lie algebra $\mathfrak{g}(G)$, the Lie algebra of the group G, which may be identified to the space of left-invariant derivations of the field of rational functions on G. The author proves that, if H is an algebraic subgroup of G, then $\mathfrak{g}(H)$ may be identified in a natural manner with a subalgebra of $\mathfrak{g}(G)$; and, if f is a "separable" algebraic homomorphism of G onto an algebraic group L, then f defines a homomorphism of $\mathfrak{g}(G)$ onto $\mathfrak{g}(L)$. *C. Chevalley* (New York, N. Y.).

Differential Geometry

Haantjes, J. Sur la géométrie infinitésimale des espaces métriques. Colloque de Géométrie Différentielle, Louvain, 1951, pp. 91–97. Georges Thone, Liège; Masson & Cie, Paris, 1951. 350 Belgian francs; 2450 French francs.

This note concerns the metrically defined non-negative function $K(p)$ which the author has attached to points p of a rectifiable metric arc to measure its curvature at p [Nederl. Akad. Wetensch., Proc. **50**, 496–508 (1947); these Rev. **9**, 53]. Denoting the Menger metrisation of curvature by $k(p)$, it is shown here again that $K(p)$ exists and equals $k(p)$ whenever the latter exists, but the existence of $K(p)$ does not imply that of $k(p)$. The new result of the paper is the determination of a large class of spaces in which the two curvatures are equivalent. This category contains all metric ptolemaic spaces (and hence euclidean space), and all riemannian spaces. *L. M. Blumenthal.*

Borsuk, Karol. Sur la notion cinématique d'une courbe. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. 41 (1948), 23–39 (1950). (French. Polish summary)

Définitions et notations: E désigne un espace topologique. Un "parcours" dans E est une application continue $x=\mathbf{x}(t)$ de l'intervalle fermé $I=[0, 1]$ dans E. Un élément y de E est une "c-valeur" de $\mathbf{x}$ si $\mathbf{x}^{-1}(y)$ contient un vrai sous-intervalle de I dit "c-intervalle" pour $\mathbf{x}$. Une fonction continue et non décroissante $\alpha(t)$ transformant I en lui-même est "compatible" avec $\mathbf{x}(t)$ si tout c-intervalle pour α est aussi un c-intervalle pour $\mathbf{x}$; la fonction α est un "régulateur" de $\mathbf{x}(t)$ lorsqu'elle est compatible avec $\mathbf{x}(t)$ et que le parcours $\mathbf{x}(\alpha^{-1}(t))$ n'a aucun c-intervalle. Deux parcours $\mathbf{x}(t)$ et $\mathbf{y}(t)$ dans E sont "équivalents", $\mathbf{x}\sim\mathbf{y}$, s'il existe deux fonctions α et β non décroissantes, transformant I en lui-même de manière continue et satisfaisant à la condition

$$(C):\quad \mathbf{x}(\alpha(t))=\mathbf{y}(\beta(t)) \text{ pour tout } t \text{ de } I.$$

Dans le cas où il existe deux fonctions α et β croissantes et satisfaisant à (C), $\mathbf{x}$ et $\mathbf{y}$ sont "fortement équivalents", $x\simeq y$.

But de l'article: Un exemple simple montre que la notion de courbe basée sur la relation d'équivalence forte n'est pas invariante vis-à-vis de la convergence uniforme. Cet article est principalement consacré à la démonstration de la transitivité de l'équivalence forte (Corollaire du Th. 2) et de son invariance vis-à-vis de la convergence uniforme.

Résultats: Th. 1: Soient f et g deux fonctions continues et non décroissantes transformant I en lui-même. Il existe deux fonctions continues non décroissantes φ et ψ transformant I en lui-même et satisfaisant à la condition $f(\varphi(t))=g(\psi(t))$ pour tout t dans I. Th. 2: Soit α_0 un régulateur du parcours $\mathbf{x}(t)$ et β_0 un régulateur du parcours $\mathbf{y}(t)$. Pour que $\mathbf{x}$ et $\mathbf{y}$ soient équivalents il faut et il suffit que $\mathbf{x}(\alpha_0^{-1}(t))$ et $\mathbf{y}(\beta_0^{-1}(t))$ soient fortement équivalents. Th. 3: Si $\{\mathbf{x}_n\}$ est une suite de parcours dans E, convergeant uniformément vers un parcours $\mathbf{x}_0$ et telle que $\mathbf{x}_n\sim\mathbf{x}_1$ pour tout $n=1, 2, \cdots$, alors $\mathbf{x}_0\sim\mathbf{x}_1$.

Remarques du référent: Les considérations du présent article se trouvent en substance dans les Thèses de M. Fréchet [Rend. Circ. Mat. Palermo 22, 1–74 (1906)] et du référent [Actualités Sci. Ind., no. 885, Hermann, Paris 1941; ces Rev. 7, 67] aux paragraphes consacrés à la représentation paramétrique des courbes. La notion d'équivalence de deux parcours coïncide avec celle du référent [loc. cit., p. 8] lorsque $\mathbf{x}$ et $\mathbf{y}$ sont envisagés comme représentations paramétriques d'une courbe d'un espace. En effet, si S est une similitude au sens large entre I et $I'=I$, soit tSt', la courbe représentative de S dans $I\times I'$ est une courbe rectifiable joignant $(0, 0)$ à $(1, 1)$; σ désignant l'arc réduit, les fonctions $t=\alpha(\sigma), t'=\beta(\sigma)$ fournissent la liaison désirée. Cette remarque contient le Th. 2. L'intervention de l'arc réduit de la courbe plane $x=f(t)$, $y=g(t)$ dans le Th. 1 conduit aussitôt au résultat. Enfin le Th. 3 est un cas particulier du Th. II du référent [loc. cit., p. 15] quand E est distancié et la distance $\delta=0$. *C. Y. Pauc* (le Cap).

***Blaschke, W. Geometría diferencial moderna. [Modern Differential Geometry].** Conferencias de Matematica, vol. I. Instituto de Matemáticas "Jorge Juan," Madrid, 1950. i+43 pp.

This is a translation of notes used by Blaschke for his expository lectures at the "Instituto Jorge Juan". Most of the subject matter is taken from Blaschke's books.

P. Scherk (Saskatoon, Sask.).

***Garnier, René. Cours de cinématique. Tome I. Cinématique du point et du solide, composition des mouvements.** 2d ed. Gauthier-Villars, Paris, 1949. iv+235 pp. 1750 francs.

Le cours que l'auteur vient de terminer est une continuation admirable des traités classiques sur le cinématique tels qu'une belle tradition française nous en a fournis depuis plus d'un siècle. Il s'agit d'un ouvrage qui considère la cinématique comme un chapitre de la mécanique rationelle et qui par là est d'un intérêt purement théorique. Les applications ne se rapportent pas en premier lieu à la science de l'ingénieur, mais aux théorèmes de la géométrie du mouvement et de la géométrie différentielle. Les expositions sont d'une clarté parfaite et quoique (au moins dans les deux premiers tomes) l'ouvrage ne vise qu'à être un manuel pratique pour les étudiants, on sent toujours que le sujet a fait le joie de l'auteur. On lit avec un vif plaisir les choses connues.

Ce tome premier expose les théories fondamentales: cinématique du point, étude du champ de vitesses et du champ d'accélération d'un solide, compositions de mouvements, détermination d'un mouvement par son champ de vitesses. La base c'est l'analyse vectorielle, combinée avec l'emploi du trièdre mobile. Relatif à la cinématique du point l'auteur donne les formules de Frenet, celles de Binet et les formules du trièdre géodésique. Pour étudier le champ des vitesses d'un solide il part du théorème énonçant que ce champ est identique à un champ de moments et il en donne deux démonstrations: l'une est une application des équations du trièdre mobile, l'autre repose sur une méthode dûe à Delassus et utilise l'ideé du champ équiprojectif. Théorèmes sur l'axe instantané et sur le mouvement hélicoïdal. Mouvements où les cônes des vitesses sont réduits à des droites. Le champ des accélérations. Composition des mouvements; le théorème de Coriolis. Mouvement d'un solide autour d'un point fixe; les angles d'Euler. Détermination du mouvement d'un solide quand le champ de vitesses est donné à tout instant. *O. Bottema* (Delft).

***Garnier, René. Cours de cinématique. Tome II. Roulement et viration, la formule de Savary et son extension à l'espace.** 2d ed. Gauthier-Villars, Paris, 1949. viii +287 pp. 2000 francs.

Ce tome a pour thème principal l'étude des questions de courbure des trajectoires et des enveloppes. Il est beaucoup moins élémentaire que le premier et il contient une série de contributions originales surtout sur le mouvement dans l'espace. L'auteur applique systématiquement la méthode du trièdre mobile, qui lui a permis d'approfondir un grand nombre de faits. Il considère en premier lieu le mouvement plan; dans les problèmes du second ordre la formule de Savary (et la construction de Bobillier) est comme on sait, le fondement de la théorie. Nous citons ici les différentes démonstrations de la formule, par exemple celle par la théorie des transformations de contact (p. 45), l'étude des trajectoires dans le voisinage du cercle des inflexions (avec le rôle remarquable du point de Ball) et la détermination des mouvements épicycloïdaux par des propriétés caractéristiques. Après avoir étendu les théories exposées au mouvement sphérique, l'auteur atteint son but principal: le mouvement dans l'espace. Dans les problèmes du premier ordre il s'agit des deux surfaces réglées, nommées axoïdes, qui jouent le rôle de la base et de la roulante de la cinématique plane et qui ont comme on sait le même paramètre de distribution pour l'axe commun; le mouvement de l'une par rapport à l'autre est en général une viration. L'auteur donne

une exposition claire sur la théorie de Koenigs concernant les courbes et les réglées associées. Quant aux problèmes du second ordre, il établit que les problèmes de courbure du mouvement spatial relèvent d'une nomographie à cinq variables, qui comprend comme cas particuliers les formules de Savary du plan et de la sphère. *O. Bottema* (Delft).

✱**Garnier, René. Cours de cinématique. Tome III. Géométrie et cinématique cayleyennes.** Gauthier-Villars, Paris, 1951. xi+376 pp. 3000 francs.

Le but du troisième tome est l'extension de la théorie cinématique à l'espace non-euclidienne. Le point de départ est un exposé parfaitement clair sur la géométrie elliptique et la géométrie hyperbolique, fondé surtout sur les idées de Cayley et Klein, mais utilisant aussi la transformation de Darboux et le demi-espace de Poincaré. Ensuite l'auteur dévéloppe la cinématique de ces espaces d'une manière systématique et complète, ce qui donne à l'oeuvre un caractère fort original. Le répère mobile a été substitué par le tétraèdre rectangle mobile, le vecteur-unité par le point normé. L'auteur en se posant toutes les questions qu'il a traitées dans la géométrie euclidienne, atteint des théorèmes intéressants et très généraux. Il ajoute deux notes dont la seconde contient un précis qui mérite d'être lu, sur la mécanique cayleyenne. *O. Bottema* (Delft).

Müller, Hans Robert. Über geschlossene Bewegungsvorgänge. Monatsh. Math. **55**, 206–214 (1951).

The paper starts with a study of closed 1-parameter families of rotations of the unit sphere of E^3, following Blaschke's treatment [S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1948, no. 2; these Rev. **11**, 132]. Applying Study's Übertragungsprinzip (in which directed lines of E^3 correspond to points on the unit sphere in 3-space with "dual" scalars, and a transformation on the directed lines, induced by a motion of E^3, corresponds to a rotation of the dual unit sphere), all the formulae are then interpreted as involving dual scalars (thus in effect considering 1-parameter families of motions of E^3). The author obtains a number of integral invariants, which are then given a geometrical meaning, by splitting the formulae into real and dual parts, and also several geometrical theorems concerning the developpables which, under the Übertragungsprinzip, correspond to the pole curves of a 1-parameter family of rotations. *H. Samelson* (Ann Arbor, Mich.).

Bilinski, Stanko. Über sphärische Evolventoiden der Raumkurven. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **6**, 106–114 (1951). (German. Serbo-Croatian summary)

Hohenberg, Fritz. Die isolierten Punkte der gestreckten Zykloiden und Trochoiden. Monatsh. Math. **55**, 242–249 (1951).

Wenn ein Kreis mit Radius r auf der x-Achse rollt, so beschreibt ein Punkt der den Abstand a vom Mittelpunkt hat, die Zykloide $x=r\varphi-a\sin\varphi$, $y=r-a\cos\varphi$. Verf. untersucht ob es für komplexe Werte $\varphi=\varphi_1+i\varphi_2$ einen reellen Kurvenpunkt geben kann; er findet die isolierten Punkte $x=rn\pi$, $y=r-a\,\mathrm{Ch}\,\varphi_2$ wenn $a<r$, n gerade, und φ_2 der Gleichung $a\,\mathrm{Sh}\,\varphi_2=r\varphi_2$ genügt. In derselben Weise werden die isolierten Punkte der (gestreckten) Trochoiden untersucht. Für die Koordinaten werden auch Näherungsformeln angegeben. *O. Bottema* (Delft).

Marussi, Antonio. Su alcune proprietà integrali delle rappresentazioni conformi di superfici su superfici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 307–310 (1951).

Let regular surface S correspond conformally to regular surface S'; i.e., $a'_{ij}=e^{2\mu}a_{ij}$ for the first fundamental tensors. Let c be a closed curve without nodes on S, c' the corresponding curve on S'; let γ be the geodesic curvature of c, γ' that of c', and write $\Gamma=\int_{(c)}\gamma ds$, $\Gamma'=\int_{(c')}\gamma' ds'$. From the theorem of Schols, namely $\gamma'=e^{-\mu}\gamma-\mathrm{grad}\,\mu\cdot n'$, where n' is the normal vector to the curve c', the author deduces that $\Gamma'-\Gamma$ is the flux of the gradient of deformation through the circuit. Then by using a Green's formula one gets the statement $\Gamma'-\Gamma=-\iint_{(A)}\mathrm{div}\,\mathrm{grad}_S\,\mu\,dS$, where A is the portion of S enclosed by c. Lastly, by referring to the Gauss-Bonnet theorem one gets the differential equation which relates the module of deformation to the total curvatures of the two surfaces: $\mathrm{div}\,\mathrm{grad}_S\,\mu=K-K'e^{2\mu}$. *A. Schwartz.*

Ničе, Vilko. Les surfaces des génératrices isotropes dans les congruences de droites du 1er ordre de la 3e, 2e et 1re classe. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **6**, 97–105 (1951). (Serbo-Croatian. French summary)

Beckert, H. Bemerkungen über die Verbiegung hyperbolisch gekrümmter Flächenstücke. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **98**, no. 4, 5–15 (1951).

The problem of determining all surfaces that can be mapped isometrically into a given surface leads to a system of quasi-linear first order differential equations for the coefficients of the second fundamental form as functions of u, v. This system is hyperbolic for surfaces with negative Gauss curvature. Known theorems about existence of solutions of such systems and their domain of dependence [see the author's paper, Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **97**, no. 5 (1950); these Rev. **12**, 415] are interpreted geometrically as statements concerning the rigidity of surfaces passing through a prescribed arc C and having prescribed tangent planes along C. *F. John.*

Lense, Josef. Projektive Kräftetransformation im R_n. Arch. Math. **2** (1949–1950), 445–448 (1951).

1) With regard to a projective transformation

$$*\xi^a=C_b{}^a\xi^b \quad (a,b,\cdots=0,1,\cdots,n)$$

the (point) Plücker coördinates Ξ^{ab} of a line $\xi\eta$ transform according to

$$*\Xi^{ab}=C_c{}^aC_d{}^b\Xi^{cd}.$$

2) Consider the differential system

$$\frac{d^2x^\nu}{dt^2}=X^\nu \quad (\nu,\lambda,\cdots=1,\cdots,n)$$

under the transformation

$$*x^\nu=\frac{C_\lambda{}^\nu x^\lambda+C_0{}^\nu}{C_\alpha{}^0x^\alpha+C_0{}^0} \quad\text{and}\quad \frac{d^*t}{dt}=\frac{k}{(C_\alpha{}^0x^\alpha+C_0{}^0)^2}$$

and write the transformed system in the form

$$\frac{d^{2*}x^\nu}{d^*t^2}=*X^\nu.$$

The main result of this paper consists of the proof (by straightforward computation) that X^ν transforms (up to a

factor) in the same way as Ξ^{n0}. The vector X^r might be identified with a force vector. *V. Hlavatý.*

Rogovol, M. R. On the projective differential geometry of nonholonomic surfaces in a three-dimensional space. Ukrain. Mat. Žurnal 2, no. 2, 102–116 (1950). (Russian)

Projective differential geometry of surfaces in 3-space has been developed in many directions. The essential purpose of this paper is to consider projective properties of non-holonomic surfaces by means of a moving trihedral. With each point $M_0(u^1, u^2, u^3)$ is associated a plane μ which is determined by a pair of points M_1, M_2 not collinear with M_0. Then a point M_3 not coplanar with μ determines a trihedral whose infinitesimal displacement is determined by four differential forms $\omega_i^k(u, du)$ $(i, k=0, 1, 2, 3)$ so that $dM_i=\omega_i^k M_k$. Any displacement of M_0 in μ is defined by $\omega_0^3=0$; if this equation is completely integrable all its integral curves passing through M_0 lie on a surface; otherwise all these curves are tangent to μ at M_0 and form a non-holonomic surface. If the differential forms ω_i^k are expressed in terms of ω_0^α by $\omega_i^k=\Gamma_{i\alpha}^k\omega_0^\alpha$, the conditions of integrability of $\omega_0^3=0$ reduce to $\Gamma_{12}^3-\Gamma_{21}^3=0$. Γ may be further specialized by choosing M_1 and M_2 on the asymptotic lines through M_0 in which case $\Gamma_{11}^3=\Gamma_{22}^3=0$. The author considers the projective correspondence of Bompiani such that M_0M_3 corresponds to M_1M_3; in this case $\Gamma_{13}^3=\Gamma_{23}^3=0$ and this correspondence is then a polarity with respect to the pencil of quadrics $z+\Gamma_{12}^3xy+\tau z^2=0$, τ being the parameter of the pencil. By considering the cross ratio of asymptotic lines the author obtains the invariant $J=-\Gamma_{12}^3/\Gamma_{21}^3$ which he defines as the projective characteristic of non-holonomy; $J=-1$ is then characteristic of the holonomic surfaces. The last section of the paper is concerned with the construction of a canonical tetrahedron by means of which the projective properties of a non-holonomic surface are expressed in terms of projective scalars. *M. S. Knebelman.*

Mikami, Misao. Projectively connected space and curves of the second order. Sūgaku (Mathematics) 1, 274–286 (1949). (Japanese)

It is well known that Weyl's projective change

$$\overline{\Gamma}_{jk}^i=\Gamma_{jk}^i+\delta_j^i\varphi_k+\delta_k^i\varphi_j \quad (i, j, k, \cdots=1, 2, \cdots, n)$$

of affine connections in the projective geometry of paths is interpreted geometrically as a change of hyperplanes at infinity for a semi-natural frame of reference in a projectively connected space [cf. K. Yano, Ann. Sci. Univ. Jassy Sect. I. 24, 395–464 (1938)]. The generalization of Weyl's projective change to an asymmetric case, that is to say, a change of asymmetric affine connections of the form $\overline{\Gamma}_{jk}^i=\Gamma_{jk}^i+\delta_j^i\varphi_k+\delta_k^i\psi_j$ was studied by V. Hlavatý [Math. Z. 28, 142–146 (1928)] and the reviewer [Proc. Imp. Acad. Tokyo 20, 284–287 (1944); these Rev. 7, 331]. The main purpose of the paper under review is to interpret geometrically this projective change of asymmetric affine connections in the geometrical schema of Cartan's theory of projectively connected spaces and to discuss some properties of the curves of the second order in such spaces.

If we adopt, in projectively connected space, a so-called semi-natural frame of reference $[A_0, A_i]$, then the connection is expressed by formulas of the form $dA_0=\omega_0^0A_0+dx^iA_i$, $dA_j=\omega_j^0A_0+\omega_j^iA_i$, or $dA_\mu=\omega_\mu^\lambda A_\lambda$ $(\lambda, \mu, \nu, \cdots=0, 1, \cdots, n)$, ω's being Pfaffian forms, and the connection is completely determined by $\omega_\mu^\lambda-\delta_\mu^\lambda\omega_0^0$. Consequently, the Pfaffian forms ω's may be subject to a transformation of the form $\tilde{\omega}_\mu^\lambda=\omega_\mu^\lambda+\delta_\mu^\lambda\varphi$, φ being an arbitrary Pfaffian form. On the other hand, if we effect a change of semi-natural frame of reference $\tilde{A}_0=A_0$, $\tilde{A}_j=\psi_jA_0+A_j$ (change of the hyperplane at infinity), the ω's will be transformed into $\tilde{\omega}$'s according to the rules $\tilde{\omega}_0^0=\omega_0^0-\psi_idx^i$, $\tilde{\omega}_j^0=\omega_j^0+d\psi_j-\psi_k\omega_j^k+\psi_j\omega_0^0-\psi_j\psi_kdx^k$, $\tilde{\omega}_j^i=\omega_j^i+\psi_jdx^i$. Combining these two transformations, the author obtains the projective change of asymmetric affine connection stated at the beginning of this review. Then he discusses torsion tensors, curvature tensors, normalized frames of reference, the determination of a projective connection corresponding to the class of affine connections projectively related to each other, associated affinely connected space (generalization of T. Y. Thomas' theory) and Cartan's projective tensor calculus. In the last half of the paper, the author studies the curves of the second order in a projectively connected space first defined by the author [Jap. J. Math. 17, 185–200 (1941); these Rev. 7, 396], H. Hombu [ibid. 17, 307–335 (1941); these Rev. 7, 396] and the reviewer and K. Takano [Proc. Imp. Acad. Tokyo 20, 410–424 (1944); these Rev. 7, 264]. The whole theory is discussed in the space of line-elements. *K. Yano.*

Dekker, David B. Generalizations of hypergeodesics. Pacific J. Math. 1, 53–57 (1951).

Hypergeodesic curves on a surface have been studied by the author in an earlier paper [Bull. Amer. Math. Soc. 55, 1151–1168 (1949); these Rev. 11, 394]. The present work is concerned with a generalization of these curves, and a discussion of a corresponding curvature and torsion which are analogous to those considered in the earlier paper.

A. Fialkow (Brooklyn, N. Y.).

Raševskiĭ, P. K. On the geometry of homogeneous spaces. Doklady Akad. Nauk SSSR (N.S.) 80, 169–171 (1951). (Russian)

A homogeneous space K_n is defined as an n-dimensional space with a (transitive) Lie group G_r. Let H_m be the stationary subgroup of G_r. All considerations are local. The algebra of Lie belonging to G_r is given by G_r', its subalgebra by H_m'. A K_n in which the Cartan metric in G_r', is non-degenerate at least on the plane H_m' is called affine-homogeneous. Then there exists in G_r' a plane E_n', $n=r-m$, which is the orthogonal complement of G_r'. With the aid of this a canonical connection is defined. When we can insert into a K_n a Riemannian metric admitting G_r, the K_n is called metrical.

Necessary and sufficient condition that a curve in an affine-homogeneous K_n be geodesic (in the sense of the canonical connection) is that it be a trajectory of the subgroup G_1 of G_r, provided that the operator X_1 of the subgroup lies in E_n' for every point M of this trajectory; for this it is sufficient that the condition holds at one arbitrary point M_0. Moreover, K_n is affine-homogeneous either if G_r be semi-simple compact or if H_m be semi-simple. Every Riemannian V_n with $ds^2>0$ admitting a transitive group of motions G_r is a metrical K_n admitting G_r. The paper ends with some remarks on the fact that the canonical connection of a K_n need not correspond to the whole G_r but to its least subgroup containing every operator X of the plane E_n'. When this canonical connection has curvature zero we obtain the symmetrical spaces of Cartan. *D. J. Struik.*

Matsumoto, Makoto. On the special Riemann spaces of class two. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. 26, 149–157 (1951).

A Riemann space V_n imbedded in a Euclidean E_{n+2} involves two "second" fundamental forms H_{ij}^1 and H_{ij}^2. If τ_I

(rank of H_{ij}^{1}) is >3 and τ_{II} (rank of H_{ij}^{II}) <2, it is known that V_n is of class one. Also if $\tau_{II}=0$, V_n is of class one. Spaces of class two may then exist for which $\tau_I=3$ and $\tau_{II}=1$. The author obtains an algebraic characterization for such spaces. This involves a set of equalities and inequalities in R_{ijkl} which assure the existence of a solution H_{ij}^{I} of the Gauss equations which has rank three. The remaining equalities and inequalities are in terms of these H_{ij}^{I} and assure the existence of H_{ij}^{II} of rank one such that these two matrices jointly satisfy the Codazzi and Ricci equations. The "type" of such a space turns out to be one, but these are not the most general spaces of class two and type one.

C. B. Allendoerfer (Seattle, Wash.).

Wintner, Aurel. On Riemann metrics of constant curvature. Amer. J. Math. **73**, 569–575 (1951).

The following theorem is proved: Let D be an open simply connected region in the space of n real variables $x_1, x_2, \cdots, x_n$ and let g_{ik} be an n-rowed real symmetric matrix of functions $g_{ik}(x_1, x_2, \cdots, x_n)$ which have a non-vanishing determinant and are of class C'. Then if no vector is changed by parallel displacement along any closed smooth curve in D, a one-to-one continuous mapping $y_i=y_i(x_1, x_2, \cdots, x_n)$ of a sufficiently small region of D exists such that the metric $g_{ik}dx_idx_k$ is equal to $h_{ik}dy_idy_k$ where the h_{ik} are constants. The essential novelty in the theorem is that no assumption of continuous second derivatives for the functions $g_{ik}(x_1, \cdots, x_n)$ is made. The result is extended to apply to (generalized) spaces of constant curvature. The proof depends upon a recent improvement of the classical existence theorem of a system of differential equations [see Hartman and Wintner, Amer. J. Math. **72**, 757–774 (1950); these Rev. **12**, 357].

A. Fialkow (Brooklyn, N. Y.).

Kuiper, Nicolaas H. Sur les propriétés conformes des espaces d'Einstein. Colloque de Géométrie Différentielle, Louvain, 1951, pp. 165–166. Georges Thone, Liège; Masson & Cie, Paris, 1951. 350 Belgian francs, 2450 French francs.

Abstract of a paper in Nederl. Akad. Wetensch., Proc. **53**, 1560–1567, 1568–1576 (1950); these Rev. **12**, 636.

Yano, Kentaro, and Sasaki, Shigeo. On the structure of spaces with normal conformal connection whose holonomy group leaves invariant a sphere of arbitrary dimension. Sūgaku (Mathematics) **1**, 18–28 (1947). (Japanese)

After reproduction of the results obtained in a previous paper [Proc. Imp. Acad. Tokyo **20**, 525–535 (1944); these Rev. **7**, 330] the authors prove the following theorems. If the holonomy group H_n of an n-dimensional space with normal conformal connection leaves invariant a sphere of $m-1$ dimensions, where $m<n$, then H_n is the direct product of two groups which are isomorphic to the Möbius groups leaving invariant a hypersphere (or a point-sphere) in Möbius spaces M^m and M^{n-m}, respectively. If and only if the holonomy group of an n-dimensional space with normal conformal connection leaves invariant $n-m+1$ mutually independent hyperspheres, the fundamental form of the space can be reduced into the form of sum of those of an m-dimensional Einstein space and of an $(n-m)$-dimensional space with constant curvature K, where the curvature scalar is equal to $m(m-1)K$.

A. Kawaguchi (Sapporo).

Craig, H. V., and Horton, C. W. On extensors and the Hamiltonian equations. Tensor N.S. **1**, 47–52 (1951).

In this paper, the authors show that the Hamiltonian equations as well as the equations of motion in general relativity can be expressed by a linear relation between three extensors. These three extensors are: (1) the momentum extensors; (2) the potential field extensors; (3) the reduced metric extensors. Essentially, the authors' idea is to write the Hamiltonian equations for generalized coordinates and to separate off that part of the derivative of the Hamiltonian which is due to the curvilinear character of the coordinate system. In this manner, one separates the Hamiltonian equations into their geometric and physical parts. The idea is applied to the equations of motion of a particle in general relativity (that is, to geodesics) and to the motion of a charged particle in special relativity.

N. Coburn.

Katsurada, Yoshie. On the non-holonomic connection of extensors. Tensor N.S. **1**, 60–66 (1951).

The purpose of this paper is to generalize the results of G. Vranceanu [Les espaces non holonomes, Mémor. Sci. Math., no. 76, Gauthier-Villars, Paris, 1936] on tensors in non-holonomic spaces to extensors in extended non-holonomic spaces. For this purpose, local base extensors are introduced and their integrability conditions examined. By this procedure, the author is led to the basic extensor of the theory. It is shown that if this extensor vanishes identically, then the original extended space is a Craig exsurface of a Euclidean space of sufficiently high dimensions. Further, the transformation law of the basic extensors under non-holonomic coordinate transformations is examined. This leads the author to a study of the non-holonomic components of extensors, and their excovariant derivatives in non-holonomic systems. Finally, the author considers non-holonomic coordinate systems in a metric exspace. It is shown that if this space admits orthogonal extensor fields for which the basic extensor vanishes then the space is a Craig exsurface.

N. Coburn (Ann Arbor, Mich.).

Tonowoka, Keinosuke. On the geometry of an $(n-1)$-ple integral of order two. Tensor N.S. **1**, 53–59 (1951).

The author considers the construction of a geometry based upon the integral

$$\int_{(n-1)} F(x^i, \partial x^i/\partial u^\alpha, \partial^2 x^i/\partial u^\alpha u^\beta)du^1du^2\cdots du^{n-1}.$$

Conditions to be satisfied by the F are given in order that the integral may be invariant under the transformations ${}^1x^a={}^1x^a(x^i)$, ${}^1u^\lambda={}^1u^\lambda(u^\alpha)$ where $a, i=1, \cdots, n$ and $\lambda, \alpha=1, \cdots, n-1$. Tensors are referred to as x-tensors or as u-tensors according as transformations of x or of u are considered. A quantity involving Greek indices is called intrinsic if it is a u-tensor with respect to all its Greek indices. The author defines generalizations of Synge vectors, coefficient of connection, covariant derivation, curvature and torsion.

E. T. Davies (Southampton).

Kawaguchi, Akitsugu. On areal spaces. II. Introduction to the theory of connections in n-dimensional spaces of the submetric class. Tensor N.S. **1**, 67–88 (1951).

The author continues his study [cf. Tensor N.S. **1**, 14–45 (1950); these Rev. **12**, 536] of the problem of constructing a geometry based on a fundamental function $F(x^i, \partial x^i/\partial u^\alpha)$ $(i=1, 2, \cdots, n; \alpha=1, 2)$ which satisfies conditions that the integral of F over a region shall be invariant. It is convenient in this study to replace the derivatives $\partial x^i/\partial u^\alpha$ by the simple

bivector p^{ij} whose components satisfy the Plücker identity. The F is then replaced by a function of x^i and p^{ij} which is homogeneous in the latter. Since the p^{ij} are not independent, it is difficult to attach a meaning to the partial derivative of the function with respect to p^{ij}. This point is considered at length in §2.

The main object of this paper, however, is to consider the case in which a two-index metric tensor can be deduced directly from the fundamental function F. This case is called the submetric case. In §1 the characteristic bitensor is introduced which provides a classification of spaces of the submetric class. In §3 a connection is introduced for the definition of parallel transport of vectors such that the length of a vector measured by the metric tensor of a submetric space does not change in transport. This connection, called the line-metric connection, does not preserve the measure of bivectors measured by the metric bitensor. Hence another connection, the area-metric connection, is introduced, which is more suited to dealing with bivectors. Torsion and curvature tensors for the two connections are introduced. *E. T. Davies* (Southampton).

Kawaguchi, Akitsugu. On areal spaces. III. The metric m-tensor in n-dimensional areal spaces based on the notion of m-dimensional area and connections in the submetric areal spaces. Tensor N.S. 1, 89–103 (1951).

In parts I and II [see the preceding review and reference there] the author has confined himself to spaces based on the notion of a 2-dimensional area. In this paper he extends some of the results to spaces based on an m-dimensional area. He again considers the submetric class, and defines connection in that case. *E. T. Davies* (Southampton).

Zhang, Ming-Yng. Die mittlere Krümmung einer Fläche im dreidimensionalen Finslerschen Raum. Acad. Sinica Science Record 3, 35–39 (1950). (German. Chinese summary)

Zhang, M. Y. Mean curvatures of a subspace in a Finsler space. Ann. Mat. Pura Appl. (4) 31, 297–302 (1950).

Generalizzando un risultato di M. Calonghi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 11, 554–558 (1930)], si dà il seguente significato per la curvatura media di un'ipersuperficie immersa in uno spazio di Finsler: Sia P un punto (regolare) di un'ipersuperficie F_{n-1} immersa in uno spazio di Finsler F_n. Si consideri in F_{n-1} una ipersuperficie F_{n-2} chiusa variabile, contenente P, e sia $w^i(Q)$ il vettore unitario (controvariante) normale ad F_{n-2} in F_{n-1} nel generico punto Q di F_{n-2}. Sia ora $w^i(Q/P)$ il vettore ottenuto spostando per parallelismo i rispetto a F_n il vettore $w^i(Q)$ da Q a P lungo un qualsiasi arco di F_{n-1}, e si consideri il complemento θ dell'angolo che $w^i(Q/P)$ forma con il vettore $n^i(P)$ normale ad F_{n-1} in F_n. La curvatura media di F_{n-1} in P, M, è allora data dal limite

$$M = \lim_{F_{n-2} \to P} \int_{F_{n-1}} \theta d\tau_{n-2} \Big/ \int_{R_{n-1}} d\tau_{n-1},$$

ove R_{n-1} indica la regione di F_{n-1} racchiusa da F_{n-2}, e $d\tau_{n-1}$ e $d\tau_{n-2}$ sono rispettivamente gli elementi di volume di R_{n-1} ed F_{n-2}. Nel primo lavoro è esaminato, con i debiti cambiamenti di parole, il caso di una superficie in uno spazio a tre dimensioni; nel secondo, che contiene anche altre generalizzazioni, si esamina il caso generale, nonchè il caso di una varietà F_m immersa in uno spazio di Finsler a più di $m+1$ dimensioni. *V. Dalla Volta* (Roma).

Galvani, O. La réalisation des connexions euclidiennes d'éléments linéaires et des espaces de Finsler. Ann. Inst. Fourier Grenoble 2 (1950), 123–146 (1951).

It is known that an analytic Riemann space can be locally imbedded in a Euclidean space of sufficiently high dimension. This paper proves the analogous result for the imbedding of a Finsler space so that each point of the Finsler space corresponds to a Euclidean configuration consisting of a point M, a line through M, and an n-plane through this line. The methods apply to even more generalized spaces such as the variational spaces of Lichnerowitz.

A linear connection $\mathfrak{L}_{2n-1}$ is defined in a space of linear elements by the differential forms: ω_i and ω_{ij}, which are assumed to be independent. If $\omega_i=0$ are completely integrable, $\mathfrak{L}_{2n-1}$ is called "semi-punctal." The points (x_i) of a variety V_n together with the "directions" (x_i') form a variety W_{2n-1}. On this is imposed a $\mathfrak{L}_{2n-1}$ such that ω_i do not depend on dx_i'. This is a space L_{2n-1}. It is semi-punctal and in it a Finsler metric can be introduced. In a Euclidean E_N $(N>n)$ we consider the multilinear element $S=(M, \Delta, P)$ where M is a point, Δ a line through M, and P an n-plane through this line. The linear element (M, Δ) is assumed to generate a variety of dimension $2n-1$. A variety of such elements S is called a V_{2n-1}. By a mapping of V_{2n-1} on E_n there is induced an $\mathfrak{L}_{2n-1}$ on V_{2n-1}. This is what is meant by a realization of the connection $\mathfrak{L}_{2n-1}$. It is proved that every analytic $\mathfrak{L}_{2n-1}$ is locally realizable in a Euclidean space where $N=2n^2-n$. If $\mathfrak{L}_{2n-1}$ is semi-punctal, it can be realized by a semi-punctal V_{2n-1} in a space where $N=2(n^2-n+1)$. The space L_{2n-1} is realizable by V_{2n-1} when $N=2n^2-n$, and by a semi-punctal V_{2n-1} in a space where $N=2(n^2-n+1)$. The last sentence holds also for a Finsler space. *C. B. Allendoerfer* (Seattle, Wash.).

Duff, G. F. D., and Spencer, D. C. Harmonic tensors on manifolds with boundary. Proc. Nat. Acad. Sci. U. S. A. 37, 614–619 (1951).

A statement of existence theorems for harmonic fields on a Riemannian space M with regular boundary B. The proofs are to appear later. Dirichlet's theorem generalises as follows: Let ψ be a p-form on B such that (i) $d\psi=0$ (on B), (ii) $\int_{\partial R}\psi=0$ $(Re H_{p+1}(M, B))$. Then there exists a unique harmonic field in M which reduces to ψ on B and has assigned periods on $R_p(M, B)$ independent relative p-cycles of M. The dual of this result generalises Neumann's theorem, and, subject to one assumption, a theorem is obtained combining the Dirichlet and Neumann theorems. *W. V. D. Hodge.*

Bochner, S. Tensorfields and Ricci curvature in Hermitian metric. Proc. Nat. Acad. Sci. U. S. A. 37, 704–706 (1951).

Let $f^{\beta_1 \cdots \beta_q}_{\alpha_1 \cdots \alpha_p}$ be a holomorphic tensor field on a compact Kaehler manifold. Then the expression

$$-\sum_{r=1}^{p} f^{\beta_1 \cdots \beta_q}_{\alpha_1 \cdots \lambda \cdots \alpha_p} R^{\lambda}_{\alpha_r \bar{\beta}_1 \cdots \bar{\beta}_q} + \sum_{s=1}^{q} f^{\beta_1 \cdots \gamma \cdots \beta_q}_{\alpha_1 \cdots \alpha_p} R^{\beta_s}_{\gamma \bar{\beta}_1 \cdots \bar{\beta}_q}$$

cannot be $\geqq 0$ at all points and >0 at some points; and if it is 0 everywhere, then the tensor must have covariant derivative 0:

$$f^{\beta_1 \cdots \beta_q}_{\alpha_1 \cdots \alpha_p, \bar{\iota}} = 0. \tag{1}$$

This assertion applies not only to holomorphic tensor fields, but, more generally, to those satisfying the equations

$$g^{\sigma\bar{\rho}} f^{\beta_1 \cdots \beta_q}_{\alpha_1 \cdots \alpha_p, \sigma\bar{\rho}} = 0. \tag{2}$$

Denote by M and m respectively the largest and smallest eigenvalues of the matrix $(-R_{\alpha\beta^*})$ formed by the components of the Ricci curvature tensor. If for $p \geqq 0$, $q \geqq 0$ we have $pm - qM \geqq 0$ everywhere and $pm - qM > 0$ somewhere, then there exists no solution of (2) for such p, q; and if we have $pm - qM = 0$, then any solution of (2) must satisfy (1). In particular, if the manifold is an Einstein space, satisfying therefore the relation $-R_{\alpha\beta^*} = bg_{\alpha\beta^*}$, then for $b > 0$ there is no holomorphic tensor with $p > q$, and for $b < 0$ none with $p < q$; and those with $p = q$ must have covariant derivative 0.

S. Chern (Chicago, Ill.).

Ehresmann, Charles. Les prolongements d'une variété différentiable. I. Calcul des jets, prolongement principal. C. R. Acad. Sci. Paris **233**, 598–600 (1951).

The author develops a theory of higher order differentials of differentiable maps; we indicate some of the definitions. If V_n, V_m are C^∞-manifolds, an r-jet (French: r-jet) of origin (Fr.: source) $x \in V_n$ and endpoint (Fr.: but) $y \in V_m$ is the class of all C^r-maps of neighborhoods of x into neighborhoods of y, with x going into y, and the partial derivatives of x, up to order r, having given values. $j_x^r f$ is the jet at x, determined by f; the function $x \to j_x^r f$ is called the r-wave (Fr.: r-flot) of f. The group $L_n^r(R^n, 0)$ of r-jets in n-space R^n, with origin and endpoint 0, and of rank n, plays an important role. An r-jet with origin 0 in R^n and endpoint $x \in V_n$ is called a p^r-velocity at x; covelocity is defined by reversing origin and endpoint; for $p = r = 1$ these are contra- and covariant vectors. An n^r-velocity of rank n in V_n is called an r-frame. The r-frames form the principal prolongement of V_n of order r. If f is a map of V_n into R^p (and if t_u is the translation of R^p which sends u into 0) then the p^r-covelocity $j_x^r(t_{f(x)} f)$ is the differential $d_x^r f$ of order r of f at x. A map of V_n into V_m induces a map of the p^r-velocities. The p^r-velocities of V_n form a fiber bundle over V_n, whose principal bundle consists of the r-frames. A theorem states that all fiber bundles associated with this principal bundle are determined by the principal prolongement of first order.

H. Samelson.

Udgaonkar, B. M. Invariants of a tensor of rank 2. Proc. Indian Acad. Sci., Sect. A. **34**, 199–200 (1951).

In this note, the author verifies by elementary methods that the number of independent rational integral invariants which can be constructed from a tensor of rank two in a metric space, with definite first fundamental form, of n-dimensions is at most n. Similarly, for an anti-symmetric tensor this number is $n/2$, or $(n-1)/2$ according to whether n is even or odd. The method of proof is based upon the theorem that any integral rational invariant of the tensor F_{jk} is a polynomial in the invariants, F^{jk}, $F^{kl}F_{kl}$, $F^{kl}F_{lr}F^{rk}$, etc. By use of the characteristic values of the tensor F_{kj} which are denoted by λ_r, the previous invariants become, $\sum\lambda_r$, $\sum\lambda_r^2$, $\sum\lambda_r^3$, etc. The theorem follows from the fact that all symmetric rational integral functions of λ_r are expressible in terms of the first n of the last invariants.

N. Coburn (Ann Arbor, Mich.).

NUMERICAL AND GRAPHICAL METHODS

✓ ***Tables of the Exponential Function e^x.** 3d ed. National Bureau of Standards Applied Mathematics Series, No. 14. U. S. Government Printing Office, Washington, D. C., 1951. xi+540 pp. $3.25.

This is the third edition of this set of tables. The first edition was published in 1939 by the Works Project Administration for the City of New York [see these Rev. **2**, 64]. The second edition was issued as MT2 of the National Bureau of Standards. Besides correcting earlier errors, this edition includes values of e and $1/e$ to 2,556 decimal places. The function e^x is listed for $x = -2.5(.0001)1$, 18D; $1(.0001)2.5(.001)5$, 15D; $5(.01)10$, 12D.

✓ ***DeLury, Daniel B. Values and Integrals of the Orthogonal Polynomials up to $n = 26$.** University of Toronto Press, Toronto, Ont., 1950. v+33 pp. $1.25.

Tables are computed for $3 \leqq n \leqq 26$. In each table the values of $\xi_r'(x, n)$ are given for $0 \leqq x \leqq n-1$, $0 \leqq r \leqq n-1$ where $\xi_r'(x, n)$ is a polynomial of degree r such that

$$\sum_{x=0}^{n-1} \xi_r'(x, n)\xi_s'(x, n) = 0$$

if $r \neq s$. Also tabulated are $\sum_{x=0}^{n-1}[\xi_r'(x, n)]^2$, $\int_0^{n-1} \xi_r'(x, n)dx$ and $\int_{-\frac{1}{2}}^{n-\frac{1}{2}} \xi_r'(x, n)dx$. These tables may be applied in the evaluation of the definite integral of a function from observed values at equidistant points. This evaluation involves approximating the function by a polynomial fitted by least squares.

H. Chernoff (Urbana, Ill.).

Abramowitz, Milton. Table of the integral,

$$\int_0^x e^{-u^2} du.$$

J. Math. Physics **30**, 162–163 (1951).

This function, useful in connection with the solution of the equation for heat conduction in the case of a fluid flowing past a flat plate, is tabulated to 8 decimals for $x = 0(0.01)2.5$; at the last argument it has reached the value of the infinite integral, namely $\frac{1}{2}\Gamma(\frac{1}{2})$ within a unit of the 8th decimal. Second differences are given.

J. C. P. Miller.

van Wijngaarden, A. Decimal-binary conversion and deconversion. Computation Dept., Math. Centrum, Amsterdam, Rep. R 130, i+41 pp. (1951).

In modern high-speed computing machines the conversion of numbers from the decimal scale into the binary, or the deconversion from the binary scale into the decimal, is performed automatically by subroutines in the program, or by built-in components of the machines. Such conversions and deconversions may, however, be desired, for example, when a machine is under construction and the available storage space does not permit such automatic computations; or when parts of the machine are working incorrectly and checking of its performance has to be done step by step.

Since the octal scale is only a shorthand-version of the binary scale, and the use of the octal scale instead of the binary scale reduces the number of digits to a third, and hence the number of steps in computation, tables with reference to this scale were prepared. No new figures have to be introduced.

The main table (pp. 6–39) gives the decimal equivalent of the octal arguments of all numbers in the range 0(8)99992. This table permits direct conversion and deconversion of integers up to 10^5. The method of applying this basic material to operations involving much larger numbers is explained by means of numerical illustrations. Table II (p. 40) gives the decimal values of 2^n and 2^{-n}, for $n = 1(1)50$; and Table III (p. 41) lists the octal values of 10^n and 10^{-n}, for $n = 1(1)18$.

R. C. Archibald (Providence, R. I.).

Siegel, K. M., Brown, D. M., Hunter, H. E., Alperin, H. A., and Quillen, C. W. The zeros of the associated Legendre functions $P^m_n(\mu')$ of non-integral degree. Engineering Research Institute, Rep. no. **UMM-82**, University of Michigan, Ann Arbor, Mich., 1951. vii+20 pp.

The authors assemble formulas for the computation of $P_n^m(\mu')$ and $Q_n^m(\mu')$ when μ' is near -1 and discuss the computation of zeros of $f(n)=P_n^m(\mu')$ when m and μ' are given. They conclude that the tables by Carrus and Treuenfels [J. Math. Physics **29**, 282–299 (1951); these Rev. **12**, 638; see also the review by Y. L. Luke in Math. Tables and Other Aids to Computation, **5**, 152–153 (1951)] are not reliable, and point out that errors in the Carrus-Treuenfels tables vitiate certain computations relating to electromagnetic waves. *A. Erdélyi* (Pasadena, Calif.).

***Gellman, Harvey. The calculation of complex hypergeometric functions with the IBM Type 602-A Calculating Punch.** Proceedings, Computation Seminar, December 1949, pp. 161–168. International Business Machines Corp., New York, N. Y., 1951.

Loud, W. S. A simple iterative solution for certain simultaneous quadratic equations. Amer. Math. Monthly **58**, 609–613 (1951).

***Lowe, John. The calculation of roots of complex polynomials using the IBM Type 602-A Calculating Punch.** Proceedings, Computation Seminar, December 1949, pp. 169–170. International Business Machines Corp., New York, N. Y., 1951.

Tasny-Tschiassny, L., and Doe, A. G. Solution of polynomial equations with the aid of the electrolytic tank. Nature **168**, 702–703 (1951).

***Kelly, John P. Matrix multiplication on the IBM Card-Programmed Electronic Calculator.** Proceedings, Computation Seminar, December 1949, pp. 47–48. International Business Machines Corp., New York, N. Y., 1951.

***Porter, Randall E. Single order reduction of a complex matrix.** Proceedings, Computation Seminar, December 1949, pp. 138–140. International Business Machines Corp., New York, N. Y., 1951.

***Brinkley, Stuart R., Jr., and Smith, Robert W., Jr. Calculation of the equilibrium composition of homogeneous multicomponent systems.** Proceedings, Seminar on Scientific Computation, November, 1949, pp. 58–63. International Business Machines Corp., New York, N. Y., 1950.

In determining the equilibrium composition of a homogeneous multicomponent gaseous system, the authors are led to the following set of simultaneous equations in the nonnegative unknowns $x_1, x_2, \cdots, x_s$: $x_j=q_j-\sum_{i=c+1}^{s}a(i,j)x_i$ $(j=1,\cdots,c)$; $x_i=k_i\prod_{j=1}^{c}x_j^{\nu(i,j)}$ $(i=c+1,\cdots,s)$. [The reviewer has abbreviated certain constants.] When the solution is such that $x_i<x_j$ for all i and j, the authors say it can be found by simple iteration of the above functions. Otherwise, the Newton-Raphson method is suggested. Suppose there exist ξ_i, η_j such that for each i there is only one j with $\nu(i,j)\xi_i\eta_j\neq 0$. Then it is possible to express all x_i, x_j in terms of two unknowns u, μ, which can be found by one of the two iterative processes. The calculations have been performed with a card-programmed calculator, but details are not given. *G. E. Forsythe* (Los Angeles, Calif.).

***Friedrich, Konrad, und Jenne, Werner. Geometrisch-anschauliche Auflösung linearer mit Nullkoeffizienten ausgestatteter Gleichungssysteme.** Deutsche Akad. Wiss. Berlin. Veröff. Geodät. Inst. Potsdam, no. **5**, viii+68 pp. (1951).

Section A contains a paper by K. Friedrich on the solution of systems of linear equations by means of geometric representations of points and continued fractions, which is closely connected with a determinant procedure. (W. Jenne states that he has given proofs of this theory without the use of continued fractions [Astr. Nachr. **278**, 73–95 (1949); these Rev. **11**, 692].) This method can be applied to problems in applied mathematics. Section B by W. Jenne is concerned with various questions related to applications of the new Friedrich determinant method and the continued fraction solution. Part I contains tables of numerical values of the determinants for the normal equations of angles of all plausible diagonal-free net configurations up to ten triangles. These tables aid in computation. Part II contains a survey of polynomials and recurrent number series of interest in various problems of solutions of equations and the theory of errors. Part III deals with the largest common factor of the elements of an adjoint determinant. This is of special use in simplifying and clarifying the computation. Part IV extends the Friedrich method of computation of all elements of an adjoint determinant and can be used to advantage with systems of equations with an unsymmetric matrix. Part V shows the use of the correlation tables in the case of unequal directional weights. Section C by K. Friedrich is concerned with a formal system which O. von Gruber developed in 1929 from the Boltz procedure [Festschr. der Techn. Hochsch. Stuttgart zur Vollendung ihres ersten Jahrhunderts 1829–1929, pp. 157–171, Springer, Berlin, 1929]. *E. Frank* (Chicago, Ill.).

***Grosh, L. E., Jr., and Usdin, E. A method for evaluating determinants and inverting matrices with arbitrary polynomial elements by IBM punched card methods.** Proceedings, Industrial Computation Seminar, September 1950, pp. 99–103. International Business Machines Corp., New York, N. Y., 1951.

Let each element of a matrix of order n be a polynomial $P_{ij}(x)=\sum_m a_m^{ij}x^m$ with literal coefficients. An IBM procedure is set up for carrying out the algebra of expressing the determinant $D=|P_{ij}(x)|$ as a polynomial in x with literal coefficients involving the $\{a_m^{ij}\}$. The equipment used is a card punch, sorter, reproducer, 602-A calculating punch, and 405 accounting machine with direct subtraction. The method used is to collect all $n!$ terms of D by powers of x. By use of three preliminary decks, the authors generate a "term deck" of at most $n!\prod_{j=1}^{n}(k_{ij}+1)$ cards, where k_{ij} $[=k_j]$ is the highest degree of the $P_{ij}(x)$ in the jth column. The coding and generation of these decks is done almost entirely by machine. The presence of zeros or duplicates among the $\{a_m^{ij}\}$ greatly decreases the size of the term deck. *G. E. Forsythe* (Los Angeles, Calif.).

***Gutshall, William D. Practical inversion of matrices of high order.** Proceedings, Computation Seminar, December 1949, pp. 171–173. International Business Machines Corp., New York, N. Y., 1951.

***Bell, William D. Punched card techniques for the solution of simultaneous equations and other matrix operations.** Proceedings, Scientific Computation Forum, 1948, pp. 28–31. International Business Machines Corp., New York, N. Y., 1950.

*Lowe, John. **Solution of simultaneous linear algebraic equations using the IBM Type 604 Electronic Calculating Punch.** Proceedings, Computation Seminar, December 1949, pp. 54–56. International Business Machines Corp., New York, N. Y., 1951.

Karush, W. An iterative method for finding characteristic vectors of a symmetric matrix. Pacific J. Math. 1, 233–248 (1951).

Let A be a real symmetric operator on real n-space. Given an initial vector $x^0 \neq 0$ and a fixed dimension s $(1<s<n)$, a sequence $\{x^i\}$ is constructed by the following gradient method. Let $\mathfrak{A}_s(x^i)$ be the subspace spanned by $x^i, Ax^i, \cdots, A^{s-1}x^i$. The vector x^{i+1} is the unique "minimum" characteristic vector, relative to the subspace $\mathfrak{A}_s(x^i)$, of the form $x^i+\eta^i$, with $(x^i, \eta^i)=0$. It is shown that x^i converges to the minimum characteristic vector in $\mathfrak{A}(x^0)$, the smallest invariant subspace containing x^0. An exact expression is given for $x^{i+1}-x^i$ in terms of certain polynomials used by Lanczos [J. Research Nat. Bur. Standards **45**, 255–282 (1950); these Rev. **13**, 163]. An analogous procedure gives the "maximum" characteristic vector. The method is compared with that of Rayleigh and Ritz in regard to practical computation. The missing part of equation (9) should read: $s<r^i$. *G. E. Forsythe* (Los Angeles, Calif.).

*Sprinkle, Leland W. **Two numerical methods of integration using predetermined factors.** Proceedings, Scientific Computation Forum, 1948, pp. 32–33. International Business Machines Corp., New York, N. Y., 1950.

Yamada, Hikoji. On a method of approximate solution of differential equations. Rep. Res. Inst. Fluid Eng. Kyushu Univ. **6**, no. 2, 42–46 (1950).

This is a short account of a longer paper in Japanese. The idea is based on the theorem of moments, so that to solve a differential equation $A(y;x)=0$ one finds a function $g(x)$ such that $\int_a^b x^n A(g(x);x)dx=0$, $n=0, 1, \cdots, N-1$. For this function the quantity $A(g(x);x)$ changes sign at least N times in (a, b). The solution for g will not in general be unique but among the g's will be one which approximates the solution of the differential equation. The method is applied to the boundary layer problem in fluid dynamics and to the progress of the freezing boundary in a liquid which is being frozen. *W. E. Milne* (Corvallis, Ore.).

Rosenberg, R. M., and Wang, A. J. Periodic solutions of a nonlinear differential equation. University of Washington. Engineering Experiment Station. Bulletin no. **118**, 77–87 (1951).

The equation $md^2x/dt^2+c|dx/dt|dx/dt+kx=F(t)$, with F periodic, is solved numerically by approximating F by a step function and using the known explicit solution [W. E. Milne, Univ. of Oregon Publ., Math. Series, vol. 2, no. 2 (1923)] for constant F. The numerical results are compared with those obtained by a method of "equivalent linearisation" [see e.g. N. Minorsky, Introduction to Non-Linear Mechanics, Edwards, Ann Arbor, Mich., 1947; these Rev. **8**, 583] and close agreement is found. *G. E. H. Reuter.*

Fröberg, Carl-Erik. On the solution of ordinary differential equations with digital computing machines. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **20**, 136–152 (1950).

The author shows how ordinary numerical integration with Simpson's rule as well as the integration of ordinary differential equations or systems of differential equations by the Runge-Kutta method can be conveniently carried out on automatic computing machines. In particular there is given a device for controlling the truncation error, by halving or doubling the interval. This makes it possible to use the best interval at each step. *W. E. Milne* (Corvallis, Ore.).

*Fenn, George S. **Programming and using the Type 603-405 Combination Machine in the solution of differential equations.** Proceedings, Scientific Computation Forum, 1948, pp. 95–98. International Business Machines Corp., New York, N. Y., 1950.

*Bisch, Paul E. **A numerical solution for systems of linear differential equations occurring in problems of structures.** Proceedings, Computation Seminar, December 1949, pp. 35–36. International Business Machines Corp., New York, N. Y., 1951.

Löwdin, Per-Olov, and Sjölander, Alf. A note on the numerical calculation of asymptotic phases with a numerical study of Hulthén's variational principle. Ark. Fys. **3**, 155–166 (1951).

The authors have computed numerical solutions of the radial wave equation of the scattering problem connected with the central field $V(r)$ which can be written in the form

$$L\psi \equiv \psi_{rr}+\{k^2-l(l+1)/r^2-c^2V(r)\}\psi=0 \tag{1}$$

for initial conditions (2) $\psi(0)=0$, $\psi_r(0)=1$, for $l=0$, $k=0.1(0.1)1.0$ and for a potential $V(r)\sim e^{-r}/r$ belonging to the deuteron problem. The solution has the asymptotic form $\psi(r)\sim A \sin(kr-\frac{1}{2}l\pi+\eta_l)$ whose phase η_l is of particular interest. The method used is a finite difference step-by-step integration: If $g(r)$ denotes the $\{\ \}$ in (1), and $\delta^{-1}g\psi$ and $\delta^{-2}g\psi$ the first and second summations of the product $g\psi$, consecutive values of these functions and of the solution ψ at arguments r_i placed h apart are computed from

$$\begin{aligned} \delta^{-1}_{i+\frac{1}{2}}g\psi &= \delta^{-1}_{i-\frac{1}{2}}g\psi+g(r_i)\psi_i, \\ \delta^{-2}_{i+1}g\psi &= \delta_i^{-2}g\psi+\delta^{-1}_{i+\frac{1}{2}}g\psi, \\ \psi_{i+1} &= \delta^{-2}_{i+1}g\psi/\{h^{-2}+g(r_{i+1})/12\}. \end{aligned} \tag{3}$$

Finite difference corrections to the approximate formulae (3) are applied subsequently, and the procedure compared with similar methods of solving linear second order equations, notably those of Fox and Goodwin [Proc. Cambridge Philos. Soc. **45**, 373–388 (1949); these Rev. **10**, 744]. In certain cases the numerical solutions are checked against accumulation of rounding off errors by integrating the nonlinear equation for the transformed wave function ω defined by $\psi=\omega \sin\chi$, $\chi_r=\alpha/\omega^2$ from large r "inwards". Finally the numerical answers are compared with the approximate solution to (1) and (2) reached by Hulthén [Ark. Mat. Astr. Fys. **35A**, no. 25 (1948); these Rev. **10**, 120], and based on a variational equivalent to (1), viz. $\int_0^\infty \psi L\{\psi\}dr=\min$, which is solved by minimising through an n-parametric family of specially chosen competition functions ψ. Surprisingly good agreement is obtained for approximate and exact phase angles η_l even for small n. *H. O. Hartley* (London).

Diaz, J. B., and Roberts, R. C. On the numerical solution of the Dirichlet problem for Laplace's difference equation. Quart. Appl. Math. **9**, 355–360 (1952).

On considère un ensemble S fini de points pris parmi les sommets d'un quadrillage régulier. On appelle point intérieur un point dont les quatre voisins sont aussi dans S. Les

autres points de S sont dits points frontières. On peut formuler un problème de Dirichlet pour l'équation aux différences:

$$4u(m, n)=u(m+1, n)+u(m-1, n)+u(m, n+1)+u(m, n-1).$$

Les auteurs montrent par des procédés calqués sur ceux de la théorie des fonctions harmoniques que ce problème a une solution unique. Ils montrent ensuite la convergence vers la solution de trois procédés. Le premier s'apparente à la méthode de balayage et est identique à une certaine version de la méthode de relaxation. Le suivant est une généralisation du premier. Le troisième consiste à poser:

$$4w_{p+1}(m, n)=w_p(m+1, n)+w_p(m-1, n)+w_p(m, n+1)+w_p(m, n-1).$$

J. Kuntzmann (Grenoble).

*Yowell, Everett C. **A Monte Carlo method of solving Laplace's equation.** Proceedings, Computation Seminar, December 1949, pp. 89–91. International Business Machines Corp., New York, N. Y., 1951.

*Polachek, Harry. **Computation of shock wave refraction on the Selective Sequence Electronic Calculator.** Proceedings, Scientific Computation Forum, 1948, pp. 107–122. International Business Machines Corp., New York, N. Y., 1950.

MacNeal, R. H., McCann, G. D., and Wilts, C. H. **The solution of aeroelastic problems by means of electrical analogies.** J. Aeronaut. Sci. 18, 777–789 (1951).

*Kraft, Hans. **Aerodynamic lattice calculations using punched cards.** Proceedings, Scientific Computation Forum, 1948, pp. 60–66. International Business Machines Corp., New York, N. Y., 1950.

*de Beauclair, W., and Sinogowitz, U. **Phasenfaktorentafel zur kristallographischen zweidimensionalen Fouriersynthese in Punkten eines Achtundvierzigstel-Netzes, enthaltend 91 Tafeln der Funktion** $\cos 2\pi hx \cos 2\pi ky$. **Erläuterung und Rechenanweisung.** Akademie-Verlag, Berlin, 1949. iii+91 pp. DM 38.

This table gives the products $\cos 2\pi hx \cos 2\pi ky$ to 5 decimals for $h, k=0(1)20, y=0/48(2/48)24/48; x=y(2/48)24/48$ on 91 large pages corresponding to the combinations of x and y each of which is arranged in 21 lines ($h=0(1)20$) and 21 columns ($k=0(1)20$). The purpose of the table is to facilitate the evaluation of expressions of the type

$$(1) \qquad \sum_h \sum_k A_{hk} \cos 2\pi hx \cos 2\pi ky$$

which is the characteristic calculation in a two-dimensional Fourier synthesis of electron density in X-ray crystallography. Transparent leaves of page format are provided on which the A_{hk} are to be written in preprinted squares. These leaves are then to be placed in position on any selected (x, y)-table of the book so that the multipliers A_{hk} appear underneath the corresponding multiplicands $\cos 2\pi hx \cos 2\pi ky$ and (1) is formed as a sum of products. This computational procedure appears to be more laborious than the repeated one-dimensional summation in accordance with

$$\sum_h \{\sum_k A_{hk} \cos 2\pi hx\} \cos 2\pi ky$$

and using single entry tables of $\cos 2\pi hx$ (say in the form of the Beevers-Lipson strips). In any case, for many purposes the tabular interval of (1/24)-th in x and y would be too wide. *H. O. Hartley* (London).

Vietoris, L. **Zum Gebrauch des harmonischen Analysators von Mader-Ott.** Z. Angew. Math. Mech. 31, 179–181 (1951).

The author describes a number of short cuts in evaluating Fourier coefficients of the form

$$a_n=\varepsilon^{-1}\int_0^{2l} f(x) \cos\{n\pi x/e\}dx;$$

$$b_n=\varepsilon^{-1}\int_0^{2l} f(x) \sin\{n\pi x/e\}dx$$

on the Mader-Ott Harmonic Analyser. Whilst with the method described by Nyström [Soc. Sci. Fenn. Comment. Phys.-Math. 9, no. 14 (1937)] $f(x)$ is drawn over the full range $0 \leqq x \leqq 2l$ and the curve follower is moved along $f(x)$ and returned along the x-axis, the author suggests that the three curves (a) $f(x)$, (b) $f(2l-x)$, (c) $-f(2l-x)$ be drawn over the first half of the range $0 \leqq x \leqq l$ only. By moving the follower along (a) and returning it via (b) or (c) it is shown how the length of the path followed can be shortened and/or the scope of the instrument be increased.

H. O. Hartley (London).

*Grems, Mandalay D. **An improved punched card method for crystal structure factor calculations.** Proceedings, Computation Seminar, December 1949, pp. 158–160. International Business Machines Corp., New York, N. Y., 1951.

Belgrano, J. **Nomographic representation of the relation:** $z=F[g(z_1, z_2), h(z_1, z_2)]$. Gaceta Mat. (1) 3, 143–152 (1951). (Spanish)

The nomogram consists of two families of straight lines for parameter z_1 and one family of curves for each of parameters z, z_2, z_3. (i) A method of successive approximations for reading unknown z_1 is developed. (ii) Given an equation $z=f(z_1, z_2, z_3)$ there is determined a pair of functions $g(z_1, z_2)$, $h(z_1, z_2)$ which are those to put the equation in the form shown in the title, whenever such representation is possible. The determination involves solving, as the case may be, one non-homogeneous or two homogeneous linear partial differential equations of the first order, each in a single unknown. (iii) A method for constructing an approximate nomogram of this type from tabular data is discussed. *J. M. Thomas* (Durham, N. C.).

Gray, Harry J., Jr. **Logical description of some digital-computer adders and counters.** Proc. I. R. E. 40, 29–33 (1952).

McCallum, D. M., and Smith, J. B. **Feedback logical computors.** Electronic Engrg. 23, 458–461 (1951).

Parks, P. C. **Building a noughts and crosses machine.** Eureka 1951, no. 14, 15–17 (1951).

*Couffignal, L. **Traits caractéristiques de la calculatrice de la machine á calculer universelle de l'Institut Blaise Pascal.** Proceedings of a Second Symposium on Large-Scale Digital Calculating Machinery, 1949, pp. 374–386. Harvard University Press, Cambridge, Mass., 1951. $8.00.

Ekelöf, Stig. Les machines mathématiques en Suède. Trans. Chalmers Univ. Tech. Gothenburg [Chalmers Tekniska Högskolas Handlingar] no. 116, 26 pp. (1951).

Auerbach, A. A., Eckert, J. P., Jr., Shaw, R. F., Weiner, J. R., and Wilson, L. D. The Binac. Proc. I. R. E. 40, 12–29 (1952).

***Hurd, Cuthbert C. The IBM Card-Programmed Electronic Calculator.** Proceedings, Seminar on Scientific Computation, November, 1949, pp. 37–41. International Business Machines Corp., New York, N. Y., 1950.

***Moore, Benjamin L. The Mark III calculator.** Proceedings of a Second Symposium on Large-Scale Digital Calculating Machinery, 1949, pp. 11–19. Harvard University Press, Cambridge, Mass., 1951. $8.00.

***Andrews, Ernest G. The Bell computer, Model VI.** Proceedings of a Second Symposium on Large-Scale Digital Calculating Machinery, 1949, pp. 20–31. Harvard University Press, Cambridge, Mass., 1951. $8.00.

***Bloch, Richard M. The Raytheon electronic digital computer.** Proceedings of a Second Symposium on Large-Scale Digital Calculating Machinery, 1949, pp. 50–64. Harvard University Press, Cambridge, Mass., 1951. $8.00.

***Lester, Burton R. A General Electric engineering digital computer.** Proceedings of a Second Symposium on Large-Scale Digital Calculating Machinery, 1949, pp. 65–70. Harvard University Press, Cambridge, Mass., 1951. $8.00.

Bubb, F. W., Jr. A circuit for generating polynomials and finding their zeros. Proc. I. R. E. 39, 1556–1561 (1951).

Artobolevskiĭ, I. I. Two mechanisms for the tracing of curves of higher order. Doklady Akad. Nauk SSSR (N.S.) 80, 717–719 (1951). (Russian)

Artobolevskiĭ, I. I. A mechanism for the solution of quadratic equations of the form $a_1x^3+a_2x+a_3=0$. Doklady Akad. Nauk SSSR (N.S.) 80, 549–551 (1951). (Russian)

ASTRONOMY

Cook, A. H. A note on the errors involved in the calculation of elevations of the geoid. Proc. Roy. Soc. London. Ser. A. 208, 133–141 (1951).

The paper is a sequel to a previous paper [same Proc. Ser. A. 204, 374–395 (1950); these Rev. 12, 542] which discussed the errors likely to occur in calculations of deflexions of the vertical from gravity anomalies. It is a discussion of the likely errors in calculations of the elevation of the geoid by Stokes' formula [Trans. Cambridge Philos. Soc. 8, 672–695 (1849)]

$$N=\frac{a}{2\pi g}\int_S \Delta g f(\psi)\sin\psi d\psi d\chi$$

(in which N is the elevation of the geoid above a given reference surface, a is the earth's radius, g the mean value of gravity, Δg the difference between observed gravity and the value corresponding to a given reference surface, $f(\psi)$ is Stokes' function, ψ and χ are the distance and azimuth respectively of Δg from the point at which N is calculated, and the integral extends over the whole earth) when the integration is restricted to a cap of radius α and the gravity data outside the cap are neglected. It is shown that by comparison with the results of Jeffreys [Monthly Notices Roy. Astr. Soc. Geophys. Suppl. 5, 55–66 (1943)] that the errors are comparable with the root-mean-square elevation. It is also shown that these estimates of errors in the elevation and likewise the estimates of errors in the deflexion of the vertical made in the earlier paper are not seriously affected by assumptions concerning the distribution of gravity anomalies between spherical harmonics up to degree 18. *J. B. Macelwane* (St. Louis, Mo.).

Clemence, G. M. Celestial mechanics. Monthly Not. Roy. Astr. Soc. 111, 219–231 (1951).

Expository paper with an extensive bibliography in a series entitled "Reports on progress of astronomy".

Popoff, Kyrille. Sur le mouvement de la Terre autour de son centre de gravité. Bull. Astr. (2) 16, 121–129 (1951).

By throwing the differential equations governing the motion of the earth about its center of gravity into integral equations of the Volterra type, the author shows that the variation in the angle of inclination of the equator to the ecliptic has no linear secular term. *R. G. Langebartel.*

Čebotarev, G. A. Application of periodic orbits to the study of the motion of small planets. Akad. Nauk SSSR. Byull. Inst. Teoret. Astr. 4, 499–554 (1951). (Russian)

Aneckstein, Jules. Un système particulier de nombres hypercomplexes et son application à l'étude des systèmes planétaires. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 459–479 (1951).

A hypercomplex number is defined as an ordered pair of numbers (a, b) satisfying certain laws of operation. It is written in the form $a+\epsilon b$, where a and b are real or ordinary complex numbers arranged in a definite order and ϵ is called the hyperimaginary unit. The two properties, the laws of addition and multiplication, which are included in the definition are:

$$(a, b)+(c, d)=(a+c, b+d),\quad (a, b)\times(c, d)=(ac+bd, ad+bc).$$

Such a system of hypercomplex numbers is characterized by the following three features: (i) the square of the hyperimaginary unit is one; (ii) for their "polar" representation in the plane determined by the axis of reals and the axis of hyperimaginaries hyperbolic functions are used; (iii) the direction of the hyperimaginary axis is reversed; i.e., it is directed from infinity toward the origin. By extension the concept of spatial hypercomplex numbers, written in the form $a+\epsilon b+ic$ $(i=\sqrt{-1})$, is introduced; geometrically they are represented by points on a sphere.

As an application, the general relation which exists between the mass and the four ecliptic elements (inclination, major semi-axis, linear eccentricity and longitude of the perihelion) of a celestial body describing an elliptic orbit is deduced. Applied to the solar system this relation provides a theoretical deduction of the angles of inclination with respect to the plane of the ecliptic of the orbits of the planets Mercury and Venus and of the mean inclination of the lunar orbit. These values have been obtained up to now by observation only. *E. Leimanis* (Vancouver, B. C.).

Chazy, Jean. Sur la valeur d'un déterminant fonctionnel. C. R. Acad. Sci. Paris **233**, 537–539 (1951).

Let x, y, z be the rectangular coordinates, x', y', z' the velocity components, of a body in elliptic motion relative to the center of attraction, and let $C_i, i=1, \cdots, 6$, be a set of six elliptic elements. The Jacobian

$$\Delta=\frac{D(x, x', y, y', z, z')}{D(C_1, \cdots, C_6)}$$

or its reciprocal arises in many problems in celestial mechanics. Its direct evaluation is laborious. The author proceeds from the equations of the variations of elements in the Lagrangian form

$$\frac{dC_i}{dt}=\sum_k P_{ik}\frac{\partial R}{\partial C_k}$$

for the conventional set of elliptic elements. The Poisson brackets P_{ik} are known in terms of the elliptic elements. Of the determinant $\|P_{ik}\|$ only ten of the 36 elements are different from zero. Since $\|P_{ik}\|=1/\Delta^2$, the evaluation of Δ, except for its sign, follows at once. *D. Brouwer.*

Danjon, André. Deux modes d'application de la méthode de Laplace pour la détermination des orbites. (Méthode des positions fictives, méthode des variations). Bull. Astr. (2) **16**, 85–110 (1951).

The essential feature of the Laplacian method of determining an orbit is the evaluation of the elements from $\alpha_0, \alpha_0', \alpha_0'', \delta_0, \delta_0', \delta_0''$, the right ascension and declination and their first and second derivatives for a single date. Three observations at times t_1, t_2, t_3 can yield only approximate values of these basic data. The usual application of the Laplacian method has been to develop procedures for correcting the results obtained in a first approximation, not for correcting the basic data. In this paper the latter approach is followed. A "method of fictitious positions" furnishes by iteration a provisional orbit that represents as accurately as one may wish three observed positions. The computations are limited to repeated applications of the same set of formulae. On account of its simplicity the method lends itself to the use of rapid calculating machines. Although in principle the method of fictitious positions can be applied to more than three positions, the author states that trials have indicated that as a rule it is better to limit the computation to three observations, leaving the others for the computation of the definitive orbit. This second phase is begun when the provisional orbit is considered to be a sufficient approximation. It is a method of differential correction that, by least-squares solutions, yields the most probable values of the basic data $\alpha_0, \alpha_0', \alpha_0'', \delta_0, \delta_0', \delta_0''$. A new series of applications of the Laplacian formulae gives the orbit for which the sum of the squares of the deviations of the observed coordinates is a minimum. One of the advantages of the Laplacian method is the possibility of introducing perturbations by one or more of the principal planets without any real complications. This aspect is not introduced. An example shows all essential details of every phase of the computations. *D. Brouwer* (New Haven, Conn.).

Vaidya, P. C. The gravitational field of a radiating star. Proc. Indian Acad. Sci., Sect. A. **33**, 264–276 (1951).

From the author's summary: A star of mass M and radius r_0 is supposed to start radiating at time t_0. The zone of radiation extends to $r=r_1$ at a later instant $t=t_1$. The energy tensor for the radiation zone, describing the directed flow of radiation, is evaluated and a relativistic line-element representing the field of radiation for $r_0 \leqq r \leqq r_1$ and corresponding to $t_0 \leqq t \leqq t_1$ is obtained. It is shown that certain quantities are conserved in the field along a world-line of flow. At $r=r_0$, $t=t_0$ and at $r=r_1$, $t=t_1$, the line element reduces to Schwarzschild's static form for a mass M. The conservation of energy is verified. *A. Schild.*

Lemaitre, Georges. Modèles mécaniques d'amas de nébuleuses. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 291–306 (1951).

The condensation theory of nebular clusters is reexamined, taking into account the velocities of the individual nebulae. In an earlier paper [same Bull. (5) **34**, 551–565 (1948); these Rev. **10**, 406] the author had considered spherically symmetric clusters. In the present work the symmetry conditions are relaxed to a more general case which the author calls "quasi-isotropic". The infinitesimal quasi-isotropic condensations are studied in detail.

A. Schild (Pittsburgh, Pa.).

Kustaanheimo, Paul. On the differential effect of the galactic rotation. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **81**, 23 pp. (1950).

Assuming rotational symmetry, galactic plane symmetry, and steady state, and that the divergence of $\rho\omega$ where ρ is the density and ω the dispersion tensor of the individual velocities vanishes in the galactic plane, the author shows that if the plane divergence of the velocity vector field is zero in the galactic plane and if ρ decreases outwards the Oort rotational model is the only one consistent with present differential rotation effect observational data. However, these assumptions (as pointed out to the author by Lindblad) imply a cylindrical distribution of matter and therefore this investigation appears not applicable to stellar systems. *R. G. Langebartel* (Urbana, Ill.).

Cowling, T. G. The condition for turbulence in rotating stars. Astrophys. J. **114**, 272–286 (1951).

In this paper the question of the extent to which the onset of convection by thermal instability can be prevented by rotation is considered. For the case of rotation with a uniform angular velocity Ω, the perturbation equations are

$$(1)\qquad -\frac{\partial}{\partial R}\delta p=\delta\rho g\sin\alpha+\rho\left(\frac{Du}{Dt}-2\Omega v\right),$$

$$(2)\qquad -\frac{\partial}{R\partial\phi}\delta p=\rho\left(\frac{Dv}{Dt}+2\Omega u\right),$$

$$(3)\qquad -\frac{\partial}{\partial z}\delta p=\delta\rho g\cos\alpha+\rho\frac{Dw}{Dt},$$

where $D/Dt=\partial/\partial t+\Omega\partial/\partial\phi$. In the foregoing equations R, ϕ and z are cylindrical polar co-ordinates with respect to a polar axis coincident with the axis of rotation, u, $v+R\Omega$ and w are the velocities of a moving element along the principal directions; further g denotes the apparent gravity (i.e., the resultant of the true gravity and the centrifugal acceleration), α is the angle which g makes with the polar axis, and δp and $\delta\rho$ are the changes in pressure and density at a particular point. In addition to equations (1)–(3) we have the equation of continuity

$$(4)\qquad \frac{D}{Dt}\delta\rho+\frac{\partial}{R\partial R}(R\rho u)+\frac{\partial}{R\partial\phi}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

and the equation

$$(5)\quad \frac{1}{p}\left(\frac{D}{Dt}\delta p+u\frac{\partial p}{\partial R}+w\frac{\partial p}{\partial z}\right)=\frac{\gamma}{\rho}\left(\frac{D}{Dt}\delta\rho+u\frac{\partial \rho}{\partial R}+w\frac{\partial \rho}{\partial z}\right),$$

which states that the changes in pressure and density following the motion take place adiabatically. (γ denotes the ratio of the specific heats.) Assuming that the volume of the convected elements are small compared with the scale of variation of the microscopic variables and writing

$$(6)\quad \frac{\partial}{\partial R}=il,\quad \frac{\partial}{R\partial\phi}=im,\quad \frac{\partial}{\partial z}=in \quad\text{and}\quad \frac{D}{Dt}=i\sigma,$$

we obtain the equations:

$$(7)\quad \rho(i\sigma u-2\Omega w)+il\delta p+g\sin\alpha\delta\rho=0,$$

$$(8)\quad \rho(i\sigma v+2\Omega u)+im\delta p=0,$$

$$(9)\quad \rho i\sigma w+in\delta p+g\cos\alpha\delta\rho=0,$$

$$(10)\quad \rho(ilu+imv+inw)+i\sigma\delta\rho=0,$$

$$(11)\quad -\rho g(1-\gamma/\Gamma)(u\sin\alpha+w\cos\alpha)+i\sigma\delta p-\gamma gHi\sigma\delta\rho=0,$$

where $H=p/g\rho$ and $\Gamma=(\rho\partial p/\partial z)/(p\partial\rho/\partial z)$. On the assumptions made, l^{-1} and n^{-1} are negligible compared to the scale height H and the solutions of the equations (7)–(11) fall into two groups: The first gives sound waves. We ignore this. The other group gives σ^2 comparable at most to g/H and Ω^2. This leads to the simplification of equations (10) and (11) to

$$(12)\quad \rho(lu+mv+nw)=0$$

and

$$(13)\quad \eta\rho(u\sin\alpha+w\cos\alpha)+i\sigma\delta\rho=0$$

where $\eta=|\text{grad } p|/\gamma p-|\text{grad }\rho|/\rho$ is the extent of the superadiabatic gradient.

The determinant of the system of equations (7)–(9), (12) and (13) is

$$(14)\quad \sigma\{\sigma^2(l^2+m^2+n^2)-4\Omega^2n^2+\eta g[m^2+(l\cos\alpha-n\sin\alpha)^2]\}=0.$$

Ignoring the root $\sigma=0$ (which gives neutral stability for certain types of horizontal motions), we shall have instability when

$$\eta g[m^2+(l\cos\alpha-n\sin\alpha)^2]>4\Omega^2n^2.$$

From this equation it follows that for $n\to 0$, rotation has no stabilizing effect. For $n=0$ the motion is such that

$$\frac{u}{m^2\sin\alpha}=-\frac{v}{lm\sin\alpha}=\frac{w}{(l^2+m^2)\cos\alpha}.$$

This is of two kinds: The first given by $m=n=0$ is a motion wholly parallel to the axis of rotation; the non-existence, in this case, of a stabilizing effect due to rotation is well known. The second given by $m=0$, $n\neq 0$ gives a "whirling motion in tubes parallel to the axis of rotation".

The case of non-uniform rotation is also considered. Discussing the problem under the same physical conditions, the author derives a second order differential equation (in R) for $U=\rho u$. For the case $n=0$ the equation in question reduces to

$$(\sigma^2+\eta g\cos^2\alpha)\frac{d^2U}{dR^2}-m^2(\sigma^2+\eta g)U=0.$$

The author deduces from this equation that "not only has the stabilizing effect of rotation disappeared, it has been replaced by an actual destabilizing effect". The author in fact proves rigorously that in some cases instability sets in already for $\eta<0$. *S. Chandrasekhar.*

MECHANICS

*Čebyšev, P. L. **Teoriya mehanizmov izvestnyh pod nazvaniem parallelogramov. S kommentariyami V. L. Gončarova, I. I. Artobolevskogo, i N. I. Levitskogo, i priloženiem stat'i I. I. Artobolevskogo i N. I. Levitskogo, Razvitie približennyh metodov sinteza mehanizmov po Čebyševu.** [**Theory of the Mechanisms Known under the Name Parallelograms. With Commentaries by V. L. Gončarov, I. I. Artobolevskiĭ, and N. I. Levitskiĭ, and the addition of an essay by I. I. Artobolevskiĭ and N. I. Levitskiĭ, The Development of Approximate Methods in the Synthesis of Mechanisms according to Čebyšev**]. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949. 80 pp.

As a contribution to the current Čebyšev revival, his 1853 paper [Théorie des mécanismes connus sous le nom de parallélogrammes, Mém. Acad. Impériale Sci. St.-Pétersbourg 7, 537–568 (1854)] has been republished (in a 1899 Russian translation [cf. these Rev. 9, 628]). ("Parallelogram" stands here for "linkage".) No specific applications to linkages are presented in the paper which is purely mathematical except for the introduction which shows the technological origin of the problem of best approximation. The principle of full oscillation (credited to Poncelet) is merely enounced, and applied to some approximation problems by means of power polynomials (34 pp.).

The commentary by Gončarov (16 pp.) is a popular lecture on the Watt linkage (with puzzling wealth of detail), followed by a mention of the credit due to Poncelet for the principle of full oscillation, a brief and restricted proof of it, and a few examples of best approximation. The commentary by I. I. Artobolevskiĭ and Levitskiĭ (16 pp.) is simply a summary of Čebyšev's papers (with some direct quotations). The appendix by the same two authors is another survey of Soviet work based on Čebyšev's approximation [see Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 4, no. 16, 5–46 (1948); these Rev. 12, 135]. All of it has been reviewed here. *A. W. Wundheiler* (Chicago, Ill.).

*Geronimus, Ya. L. **O primenenii metodov Čebyševa k zadače uravnovešivaniya mehanizmov.** [**On the Application of Čebyšev's Methods to the Problem of the Balancing of Mechanisms**]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1948. 148 pp.

The first chapter (52 pp.) of the book presents a survey of some approximation methods (interpolation, expansions, least-squares, Čebyšev's uniform approximations combined with successive ones) with special attention to approximate expressions of the form $\sum c_k\zeta_k$, $\zeta_k=\exp(ik\alpha)$ with complex c_k. A graphical method is given for the last mentioned method. The remainder is an expansion and an elaboration of the author's previously published ideas [Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 4, no. 14, 64–67 (1948); 2, 164–174 (1947); these Rev. 13, 170]. Mechanisms containing members rotating about axes perpendicular to the crankshaft are considered.

The mathematical feature of the method is the use of complex numbers to represent the main inertial force p' and the transverse inertial moment m'_{xy} of the crankshaft: $p'=R\omega^2\sum c_k\zeta_k$, $m'_{xy}=R\omega^2\sum d_k\zeta_k$. The axial inertial moment is real: $M_z'=R\omega^2\sum(g_k\cos k\alpha_k+h_k\sin k\alpha)$. The origin of the moments is the center of symmetry of the shaft, α is the crank angle, ω the angular speed. The counterweights of masses m are assumed to rotate about axes parallel to the crankshaft. Their inertial main forces can be represented by $\mu R\omega^2\zeta$, and inertial moments by $i\mu zR\omega^2\zeta$, where $\mu=m\exp(i\phi)$; R, ϕ, z are the cylindrical coordinates of the counterweights. The author distinguishes between "outer" balancing when only the total main force and the total inertial torque are separately minimized (in the least-squares or the Čebyšev sense) and "full" balancing when each of the bearing reactions is minimized.

Chapter 2 (27 pp.) is devoted to the problem of outer balancing by means of two counterweights. Both least-squares and Čebyšev minimizing are considered. If the counterweights are solid with the shaft, only transverse balancing can be achieved. If they are made to rotate about axes parallel to the shaft, axial minimal balancing can be achieved as well. The results are extended to the case in which each throw of the crankshaft controls the same mechanism (as in engines of the types V, W, or X). It is shown that the solution of this problem can be easily derived from that of the balancing of the crankshaft alone, by multiplying the masses of the counterweights by the same factor, and turning them by the same angle. A specific application is made to the eccentric of a locomotive.

Chapter 3 (20 pp.) deals with full balancing. There arises here the additional problem of the distribution of the reactions among the bearings, and influence coefficients are determined. Chapter 4 (12 pp.) concentrates on the one-cylinder reciprocating engine. Chapter 5 (16 pp.) considers bearing reactions only (see the second paper cited above) assuming negligible resultant inertia forces. Minima are required for each reaction and for the total mass of the counterweights. Here least-squares minima are used, and gas reactions accounted for.

The novelty of the author's approach consists in going beyond the conventional method of balancing the first or second harmonics (in the crank angle) of the inertial forces.

A. W. Wundheiler (Chicago, Ill.).

Semenov, M. V. Design of partly balanced plane mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **8**, no. 29, 74–90 (1949). (Russian)

The author considers the balancing of the first harmonic of the resultant inertia of a plane mechanism, by modifying only the mass distribution of some working members. For a four-bar or slider-crank linkage in one plane he does it by moving the mass centers of the cranks, or of one crank and the connecting rod. Other examples are considered before mechanisms having several parallel planes are considered. The equations are simple and fully treated.

A. W. Wundheiler (Chicago, Ill.).

Semenov, M. V. Balancing of spatial mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **8**, no. 32, 31–42 (1950). (Russian)

It is shown, for rotating machinery in space, that if the 12 coefficients of the kth harmonic of the inertia force are given, three balancing counterweights can be easily determined (they must be mounted on conical gears rotating at the speed $k\omega$). For the case of the first harmonic, the author extends the method of the preceding paper to the spatial case.

A. W. Wundheiler (Chicago, Ill.).

Abramov, B. M. An approximate method for the investigation of the motion of mechanisms, taking account of frictional forces. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **6**, no. 22, 5–27 (1949). (Russian)

When exact consideration of friction is impractical, the common expedient is to assume that the normal reactions are the same as the smooth ones and apply the conventional laws of dry friction. The paper reviews this procedure for systems of one degree of freedom, and maps it out for a five-bar linkage as an example of two degrees of freedom. Lagrange equations are used after the generalized friction forces are determined. Some closer attention is given to the case of a two-bar three-hinge component of a linkage. No insights into the phenomena are given.

A. W. Wundheiler (Chicago, Ill.).

Artobolevskiĭ, S. I. Foundations of the kinematics of automatic machines. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **1**, 81–126 (1947). (Russian)

The author introduces some terminology and some logic for the discussion of the general design of automatic machinery. His concern is the timing and duration of the component processes involved. For reciprocating parts the dependence of the duration of the cycle on the law of acceleration is discussed. Some routines (graphical and others) for the setting up of a timing plan and of activating elements are described, also in the case of several driving shafts.

A. W. Wundheiler (Chicago, Ill.).

Tallqvist, Hj. Generalized connecting-rod motion. Soc. Sci. Fenn. Comment. Phys.-Math. **14**, no. 16, 14 pp. (1950). (Swedish).

An introductory discussion of the sliding-block linkage with offset: types of block motion, centrodes, the equation of the connecting-rod curves, and some shape discussion. The last two pages indicate the steps to follow in treating the same problems for a four-bar linkage.

A. W. Wundheiler (Chicago, Ill.).

Steward, G. C. On the cardinal points in plane kinematics. Philos. Trans. Roy. Soc. London. Ser. A. **244**, 19–46 (1951).

The classical geometry of motion in a plane, which had been developed by Chebyshef, Burmester, Müller and others, is derived here by means of the complex variable. One set of cardinal points are the instantaneous centers of rotation, velocity, acceleration and higher derivatives in the fixed plane; a second corresponding set exists in the moving plane body. Relations between them are derived as well as their relations to the Ball point, the inflection circle, the centers of curvature of the paths and the evolutes, the Burmester points, the Burmester cubic and quartic curves and their dual curves. Applications are made to such special cases as four-bar linkage motions and elliptical and conchoidal displacements. Similar investigations have been made in recent papers. In particular, see Bereis [Österreich. Ing.-Arch. **5**, 246–266 (1951); these Rev. **13**, 292] and van der Woude [Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 117–122 (1951); these Rev. **12**, 867].

M. Goldberg (Washington, D. C.).

Colombo, Giuseppe. Osservazioni sulla stabilità dei moti merostatici di un giroscopio ed applicazioni ad un caso notevole. Rend. Sem. Mat. Univ. Padova **20**, 59–77 (1951).

Hypotheses on the existence of merostatic motions of a gyroscope (i.e. regular precessions or uniform rotations) lead to certain forms for the moment M of external forces. Further conditions on M that the motion should be stable are studied and application is made to the problem of a gyroscope carrying an electric charge distribution and immersed in a constant magnetic field. Cases of both stability and instability are encountered in this problem.

D. C. Lewis (Baltimore, Md.).

Colombo, G. Osservazioni ed aggiunte ad una nota precedente. Rend. Sem. Mat. Univ. Padova **20**, 219–223 (1951).

Part of the paper reviewed above is clarified in a more general setting. The author considers conservative holonomic dynamical systems with k ignorable coordinates $p_1, \cdots, p_k$ and n other coordinates $q_1, \cdots, q_n$. The merostatic motions are those in which the $\dot{q}$'s are zero and the $\dot{p}$'s are constants. A theory of stability is obtained generalizing that for ordinary equilibrium, according to which stability occurs where the potential energy is a minimum.

D. C. Lewis (Baltimore, Md.).

Neĭmark, Yu. I., and Fufaev, N. A. On an error of V. Volterra in his derivation of the equations of motion of a nonholonomic system. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 642–648 (1951). (Russian)

Volterra's equations,

$$d(\partial T/\partial \dot{p}_s)/dt = \sum a'_{sk}(\partial T/\partial p_r)\dot{p}_k + T_s + P_s$$

[Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. **33**, 451–475 (1898)] for nonholonomic systems, are correct but their derivation is based on the relation $d\delta x_i - \delta dx_i = 0$ which, in general, does not hold for holonomic systems. [This error was made and noticed by many writers.] The major part of the paper deals with errors in a paper of V. V. Dobronravov [Učenye Zapiski Moskov. Gos. Univ. Mehanika **122**, tom II, 77–182 (1948); these Rev. **10**, 630] who, from the same wrong relation, derived a wrong form of nonholonomic equations and a wrong generalization of the Hamilton-Jacobi theorem. These are disproved by means of examples. The authors seem to blame all this on Volterra.

A. W. Wundheiler (Chicago, Ill.).

Tietz, Horst. Die klassische Mechanik als Transformationstheorie. Z. Naturforschung **6a**, 417–420 (1951).

It is well-known that the Hamiltonian theory can be completely described by canonical transformations. The author shows this again. This treatment is short and clear.

J. Haantjes (Leiden).

Wilker, P. Zur Homogenisierung des kanonischen Formalismus. Z. Physik **130**, 245–255 (1951).

A formulation of classical mechanics is presented in which the time is treated as a generalized coordinate on the same basis as all other coordinates. Lagrangian and Hamiltonian functions are constructed for the case that any one of the generalized coordinates is singled out and treated as is conventionally done with "t". The relations between the various formulations are discussed. Contact transformations and the Hamilton-Jacobi equations in this symmetrical formulation are considered. The author seems unaware of the closely related work of Dirac [Proc. Cambridge Philos. Soc. **29**, 389–400 (1933)].

K. M. Case.

Werfeli, A., und Wilker, P. Über mechanische und relativistische Erhaltungssätze. Z. Physik **130**, 256–258 (1951).

Starting from the homogeneous Lagrangian formulation of mechanics of Wilker [see the preceding review], the conventional conservation laws are derived from considerations of the invariance of the Lagrangian under various infinitesimal transformations.

K. M. Case.

Hydrodynamics, Aerodynamics, Acoustics

Truesdell, C. A new definition of a fluid. II. The Maxwellian fluid. J. Math. Pures Appl. (9) **30**, 111–158 (1951).

Après avoir rappelé l'essentiel de sa théorie axiomatique du fluide de Stokes le plus général [le même J. **29**, 215–244 (1950); ces Rev. **12**, 761], l'auteur note que les concepts ainsi introduits sont impuissants à rendre compte de certains phénomènes dont les fluides réels sont le siège: l'état de tensions, mouvement prenant naissance sous l'action d'un gradient de la température, etc. L'auteur est alors amené à modifier ses schémas et à construire une théorie absolument générale du fluide continu et anisotrope, susceptible de fournir une explication théorique des faits ci-dessus rappelés.

À cet effet, l'auteur admet que les composantes t_j^i du tenseur des efforts internes, d'une part, les composantes q_i du flux calorifique, d'autre part, sont des fonctions: des composantes du tenseur des déformations, des dérivées partielles d'ordre quelconque de ces composantes par rapport aux coordonnées spatiales; des variables d'état et des gradients d'ordre quelconque de celles-ci. Les t_j^i et les q_i sont supposées développables en série de Taylor par rapport à tous leurs arguments. L'analyse dimensionnelle permet alors de préciser la forme des coefficients des développements.

On conçoit que malgré quelques résultats partiels d'une simplicité inattendue, l'ensemble des formules obtenues par l'auteur soit d'une grande complexité. En se limitant même aux corps isotropes, on ne peut obtenir, comme le montre l'auteur, les réductions des termes que l'on observe dans le cas du fluide de Stokes. L'auteur se borne donc à calculer les trois premiers termes des développements des t_j^i, ordonnés suivant les puissances de la viscosité naturelle μ_n. De plus (et contrairement à ce qui se passe pour le fluide de Stokes) on ne peut trouver ici de paramètre numérique dont la donnée permettrait d'apprécier a priori l'importance relative des divers termes des développements.

Dans la théorie classique de la viscosité, la fonction de dissipation Φ est assujettie a priori à être une forme définie positive des coefficients de déformation. Cette condition se traduit par les inégalités bien connues: $\mu > 0$; $3\lambda + 2\mu > 0$. Au contraire, l'extrême complexité des formules de l'auteur ne permet guère la discussion du signe des termes du développement de Φ ordonné suivant les puissances de μ_n.

Pour finir, l'auteur compare ses conclusions à celles que donne la théorie cinétique des gaz. Burnett [Proc. London Math. Soc. **40**, 382–435 (1936)] a déduit de cette théorie les expressions des t_j^i; Chapman et Cowling en ont tiré les valeurs des q_i [cf. The mathematical theory of non-uniform gases, Cambridge, 1939; ces Rev. **1**, 187]. Il se trouve que les méthodes statistiques ne conduisent pas aux développe-

ments aussi généraux que ceux de l'auteur et ne permettent pas de rendre compte de tous les phénomènes observés et compatibles—qualitativement, du moins—avec les équations générales obtenues dans le travail analysé (comme, par exemple, le phénomène de M. Brillouin, qui consiste en ce qu'aux pressions assez faibles et aux températures élevées un flux de chaleur prend naissance au sein du fluide sous l'action d'un gradient de la pression assez grand et en l'absence de tout gradient thermique). Cependant, l'auteur met en lumière certains avantages propres aux méthodes statistiques, dont le développement doit se poursuivre parallèlement à celui des conceptions exposées dans ce travail.

J. Kravtchenko et *R. Gerber* (Grenoble).

Vallander, S. V. The equations of motion of a viscous gas. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 409–432 (1951). (Russian)

L'auteur analyse les phénomènes moléculaires de transport (de masse, de chaleur, de quantité de mouvement) au sein d'une gaz visqueux, assimilé, au point de vue thermodynamique, à un gaz parfait. Pour certaines classes d'écoulements, les processus microscopiques auraient, à l'échelle macroscopique, une influence sensible négligée, cependant, par les théories classiques. L'auteur est ainsi amené à reviser les équations qui gouvernent l'évolution thermomécanique du gaz; au système classique d'équations aux dérivées partielles, il propose de substituer un système de cinq relations aux dérivées partielles entre les composantes de la vitesse moyenne locale, la densité et la température (système qu'il complète par des conditions aux limites nouvelles). Chemin faisant, l'auteur est amené à introduire axiomatiquement plusieurs paramètres nouveaux, caractérisant le gaz, dont il paraît difficile de déterminer les valeurs numériques à partir des mesures expérimentales dont on dispose. Deux exemples d'intégration approchées du système proposé par l'auteur achèvent le mémoire. *J. Kravtchenko* (Grenoble).

Gotusso, Guido. Un principio variazionale in idrodinamica piana. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 130–132 (1951).

The paper deals with steady plane rotational motion of a perfect incompressible fluid. It is known that the stream function ψ of such a motion satisfies $\Delta\psi = -F'(\psi)$, where F is some function and F' its derivative. The author proposes to call $K = \rho\int_\sigma F d\sigma$ the complementary energy (ρ = density, σ = area occupied by the fluid). He shows that $T - K$, where T is the kinetic energy, has a stationary value for a natural motion when compared with varied motions for which the function F is the same and ψ is unvaried on the boundary.

J. L. Synge (Dublin).

Woods, L. C. A new relaxation treatment of flow with axial symmetry. Quart. J. Mech. Appl. Math. **4**, 358–370 (1951).

The equations of motion for incompressible, irrotational flow are written with the potential ϕ and Stokes' stream function ψ as independent and the cylindrical coordinates x and r as dependent variables. The second order equation $(f_\phi/f)_\phi + f_{\psi\psi} = 0$ in $f = r^2$ alone is obtained. This equation is approximated by difference equations on a rectangular grid in the (ϕ, ψ)-plane. These, in turn, are solved by iterative methods, starting with the solution of the linearized problem as a first approximation. From f it is possible to calculate directly all the desired properties of the flow. In the case of the design of ducts, the region of flow in the (ϕ, ψ)-plane lies between the two straight lines $\psi = \psi_1$ and $\psi = \psi_2$; and for boundary conditions r is known as a function of x along these lines. In order to carry out the solution of the above difference equations, the boundary values of r must be expressed in terms of ϕ instead of x. This can be accomplished by a second iterative procedure in which a first approximate solution is used to express x in terms of ϕ and this, in turn, to give a better solution, etc. Prescribed values of the speed q on the boundary may be satisfied similarly.

Subsonic compressible flow is attacked by solving the incompressible problem first and then applying a perturbation method. In this case the independent variable $L = \log 1/q$ is employed, and the angular difference between the directions of compressible and incompressible flows is neglected. An application of the method is made to the design of the contraction section of a wind tunnel with a linear pressure gradient along the wall. *P. W. Ketchum* (Urbana, Ill.).

Katzoff, S., Gardner, Clifford S., Diesendruck, Leo, and Eisenstadt, Bertram J. Linear theory of boundary effects in open wind tunnels with finite jet lengths. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **976**, 37 pp. (1950).

Prandtl's classical theory of boundary corrections for open wind tunnels disregards the closed entrance and exit sections and is based on construction of an incompressible perturbation velocity potential which is constant on the entire infinite cylindrical boundary of the open jet. To take into account the effects of closed entrance and exit sections the authors impose the following boundary conditions. 1) On the open part of the cylindrical boundary the perturbation velocity component parallel to the tunnel axis vanishes. 2) The velocity is continuous at the lip of the entrance to the open jet, an analog of the Kutta condition. 3) The velocities at infinity in the closed sections are equal, to prevent expansion or contraction of the jet. The authors consider plane flows about a vortex in a channel with a closed section followed by an infinite open jet; a finite open jet followed by a jet closed on one side; a finite open jet followed by a closed jet; or the preceding configuration with unequal pressures on the boundaries of the open jet. Complex velocities can easily be found for all cases after conformally mapping the channel onto a half plane. For a circular tunnel the authors modify the potential for a horseshoe vortex in a closed tunnel exhibited by Eisenstadt [same Tech. Notes, no. **1265** (1947)]. The additional potential required for a finite open section is expanded into a series of Bessel functions. Numerical examples for two-dimensional and circular tunnels indicate that the interference velocities at a lifting element more than half the tunnel height downstream from the entrance are practically the same as for an infinitely long open tunnel. Also discussed are electrical analogies for experimental investigation of such flows. *J. H. Giese* (Havre de Grace, Md.).

Birkhoff, G., Plesset, M., and Simmons, N. Wall effects in cavity flow. II. Quart. Appl. Math. **9**, 413–421 (1952).

The authors extend their previous study of wall effects in cavity flows [same Quart. **8**, 151–168 (1950); these Rev. **12**, 297] to include the case of finite cavities. Their calculations are based on the Riabouchinsky model of the finite cavity in which two plates of the same width are disposed symmetrically athwart an incident parallel flow while two free streamlines joining the upper and lower edges enclose with the plates a constant pressure cavity. For such cavities in an unbounded stream, in a channel of finite width, and in a

free jet of finite width, the standard conformal mapping technique based on Schwarz-Christoffel is applied by the authors to obtain closed formulas in terms of elliptic functions for the velocity field, cavity shape, and drag coefficient as functions of cavitation number. (In this model the cavitation number is varied by changing the plate distance.) They observe that for fixed cavitation number there is a maximum "blockage" ratio of plate to channel width, this maximum corresponding to the infinite cavity of part I [loc. cit]. Thus for plates 1/20 the width of the channel, they calculate that the minimum attainable cavitation number is the (unexpectedly high) value of 0.6. *D. Gilbarg.*

Davies, T. V. The theory of symmetrical gravity waves of finite amplitude. I. Proc. Roy. Soc. London. Ser. A. **208**, 475–486 (1951).

Levi-Cività [Math. Ann. **93**, 264–314 (1925)] reduced the problem of determining the existence of two-dimensional, symmetric, periodic gravity waves of finite amplitude on an infinitely deep fluid to that of finding a function $\omega(f)$, $\omega=\vartheta+i\tau$, $f=\varphi+i\psi$, analytic in the half-strip bounded by $\varphi=\pm\frac{1}{2}c\lambda$, $\psi=0$, and $\psi=-\infty$, such that $\omega(\pm\frac{1}{2}c\lambda)=0$, $\lim_{\psi\to-\infty}\omega(f)=0$, and $d\tau/d\varphi=gc^{-3}e^{-3\tau}\sin\vartheta$ on $\psi=0$. (Here φ and ψ are the potential and stream functions, the complex velocity is $w=u-iv=ce^{-i\omega}$, and λ and c are the wave length and velocity.) Levi-Cività expanded the nonlinear boundary condition in a power series of which the first approximation, $d\tau/d\varphi=gc^{-3}\vartheta$, gives the classical linearized theory of waves. However, retention of higher order terms in this series does not aid very much in studying such phenomena as the breaking of waves. The author expands $\sin\vartheta$ in powers of $\sin 3\vartheta$, so that the first approximation, $d\tau/d\varphi=\frac{1}{3}gc^{-3}e^{-3\tau}\sin 3\vartheta$, is again a nonlinear boundary condition, but one for which the problem can be solved explicitly. The solution to this problem exhibits qualitatively the behavior of waves in the whole range from infinitesimal waves to breaking waves, depending upon the value of a parameter A in the solution. Retention of higher order terms now allows a more precise investigation of this range. The author carries through pertinent computations retaining terms through the 5th power: e.g., the relation between c, λ, and A; the form of the free surface; and the ratio of the amplitude to λ.

J. V. Wehausen (Providence, R. I.).

Sekerž-Zen'kovič, Ya. I. Composite standing waves of finite amplitude on the surface of a heavy liquid of infinite depth. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. **1951**, no. 5, 68–83 (1951). (Russian)

In earlier papers [Doklady Akad. Nauk SSSR (N.S.) **58**, 551–553 (1947); Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **15**, 57–73 (1951); these Rev. **10**, 646; **12**, 870] the author has treated gravity waves of finite amplitude by using Lagrangian coordinates and expanding in a power series in a parameter. In the present paper he expands in a double series in two parameters in such a way that the linear terms give the sum of two sinusoidal (i.e. linearized) waves. Computation of the coefficients is given through the third power of the parameters. The form of the free surface is then given explicitly, retaining terms of the second order. Perhaps the most striking feature of the solution is the presence of "difference" and "summation" waves representing an interaction between the linearized waves. As in the earlier papers, there are no fixed nodes and the free surface is never completely flat. *J. V. Wehausen* (Providence, R. I.).

***Stoker, J. J. Mathematical theory of water waves.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 304–307. Amer. Math. Soc., Providence, R. I., 1952.

A brief nonmathematical description of the development of the theory of water waves together with a bibliography of some of the more important papers of recent years. [Reviewer's remark. It seems to the reviewer that even in such a brief summary and admittedly incomplete bibliography the work of such mathematicians as Haskind, Kočin, Lavrent'ev, Nekrasov, Sretenskiĭ and others of the U.S.S.R. should not have been passed over in complete silence. A description of this work with a rather complete bibliography (of exclusively Russian work) may be found in Sretenskiĭ's article in Mehanika v SSSR za tridcat' let, Moscow-Leningrad, 1950, pp. 279–299; cf. these Rev. **12**, 660.]

J. V. Wehausen (Providence, R. I.).

Dorrestein, R. General linearized theory of the effect of surface films on water ripples. I, II. Nederl. Akad. Wetensch. Proc. Ser. B. **54**, 260–272, 350–356 (1951).

The author generalizes the discussion in Lamb's Hydrodynamics [6th ed., Cambridge Univ. Press, 1932, §349] to give a linearized theory of gravity waves which includes viscosity, surface viscosity, and surface tension. The treatment of surface tension allows for compressibility of the surface film (i.e. a sort of two-dimensional gas law) and hysteresis. The method follows closely that given by Lamb, where only viscosity and constant surface tension is considered. Some relevant tables, pertaining chiefly to water, are included. *J. V. Wehausen* (Providence, R. I.).

Tomotika, S., Tamada, K., and Umemoto, H. The lift and moment acting on a circular-arc aerofoil in a stream bounded by a plane wall. Quart. J. Mech. Appl. Math. **4**, 1–22 (1951).

The lift and moment of an aerofoil in the form of a circular arc placed in a stream bounded by a plane wall are discussed. The earlier work of Green [Proc. London Math. Soc. (2) **46**, 19–54 (1939); these Rev. **1**, 90] and Hudimoto [Mem. Coll. Eng. Kyoto Imp. Univ. **8**, 36–41 (1934)] on this problem is reviewed and by adopting a method similar to that of Hudimoto it is found that when the camber of the aerofoil is small the effect of the wall on the lift and moment is similar to that for an aerofoil in the form of a flat plate, namely, as the arc-aerofoil approaches the wall the lift and moment coefficients first decrease and then increase to values which are greater than the corresponding values for an arc-aerofoil in an unlimited stream.

From the author's summary.

Tomotika, S., Hasimoto, Z., and Urano, K. The forces acting on an aerofoil of approximate Joukowski type in a stream bounded by a plane wall. Quart. J. Mech. Appl. Math. **4**, 289–307 (1951).

In this paper the modifications introduced by the thickness of an aerofoil on the ground-effect upon its lift are discussed. The authors first derive the complex velocity function for the steady two-dimensional continuous potential flow of an incompressible inviscid fluid past two equal circular cylinders, under the assumption that the line joining their centres is perpendicular to the direction of the undisturbed flow at infinity, in terms of Weierstrass's $\wp$-function with periods depending on the relative position of the two cylinders and the common radius. A circulation of appropriate strength to give stagnation points on the

boundaries of the cylinders is also introduced. By means of a conformal transformation the region outside the two cylinders is then transformed to the region exterior to two equal aerofoils in the z-plane which are symmetrically situated with respect to the x-axis which is the direction of the flow at infinity in the z-plane. In the field of flow past the aerofoils thus obtained, the axis of symmetry is a stream line and can then be replaced by a plane rigid wall. The lift on the aerofoil in the neighbourhood of this wall is then determined.

The thickness, camber, condition that the sharp trailing edge of the aerofoil should be the stagnation point, depend on the values of certain constants in the transformation, and by varying these, aerofoils of slightly different thickness and camber are compared. It is found that when an aerofoil is placed in the vicinity of the ground at the small angles of attack used in practice, the lift on the aerofoil is in general increased by the presence of the ground, but the rate of increase of the lift becomes smaller as the thickness of the aerofoil increases. It is also shown that the effect on the results for a plane aerofoil due to thickness acts in exactly the same direction as the effect due to camber. This latter result is noted to conflict with the corresponding result given by Green [Quart. J. Math., Oxford Ser. 18, 167–177 (1947); these Rev. 9, 113]. *R. M. Morris* (Cardiff).

Tomotika, S., and Aoi, T. The pressure distributions on the surface of an obstacle in a running viscous fluid at small Reynolds numbers. Mem. Coll. Sci. Univ. Kyoto Ser. A. 26, 9–19 (1950).

Continuing their earlier study [Quart. J. Mech. Appl. Math. 3, 140–161 (1950); these Rev. 12, 59] of the exact solution of Oseen's equation for the slow flow about a sphere and a circular cylinder, the authors compute the pressure distributions about these bodies. Numerical results are presented for Reynolds numbers (based on diameter) 1 and 2 for the sphere and 0.8 and 4 for the cylinder.

J. V. Wehausen (Providence, R. I.).

Slezkin, N. A. On the differential equations of filtration. Doklady Akad. Nauk SSSR (N.S.) 79, 755–758 (1951). (Russian)

Equations expressing conservation of mass and energy and Newton's second law are derived for motion within a mixture of a fluid and a mobile solid. Specialization of the equations to apply to incompressible materials, immobile solids and other cases are made, but no specific problems are set up or solved. *R. E. Gaskell* (Seattle, Wash.).

Polubarinova-Kočina, P. Ya. On the dynamics of ground water with sprinkling. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 649–654 (1951). (Russian)

In this paper there is considered the outflow of ground water from a mound of groundwater produced by the uniform sprinkling of a bounded region of a pervious layer of soil supported by an impervious layer. The results consist in a graphic description of the known explicit solution $u(x, t)$ of the differential equation $u_t = a^2 u_{xx} + f(x, t)$ under various boundary conditions. *H. P. Thielman* (Ames, Iowa).

Galin, L. A. Some problems of unsteady motion of ground water. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 655–678 (1951). (Russian)

The problems considered deal with the outflow of water from mounds of ground water in layers of infinite or finite depths. The solution gives the height of the free surface of the water, and the velocity potentials in term of complex integrals of the Cauchy type. *H. P. Thielman.*

Kočina, N. N. The plane problem of the outflow of a mound of ground water in a layer of infinite depth. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 679–682 (1951). (Russian)

This is another treatment of one of the problems considered by L. A. Galin [see the preceding review]. The method used here is an adaptation of one used by N. E. Kočin [Sobranie sočineniĭ (Collected works), v. 2. pp. 277–304, Moscow-Leningrad, 1949; these Rev. 12, 59] in the treatment of an analogous problem in the theory of waves on the surface of an incompressible fluid.

H. P. Thielman (Ames, Iowa).

Sokolov, Yu. D. On the flow of ground water into a drainage ditch of trapezoidal section. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 683–688 (1951). (Russian)

This is an extension and correction of an earlier treatment of the same problem [Polubarinova-Kočina and Fal'kovič, same journal 11, 629–674 (1947); these Rev. 10, 73; translated in Advances in applied mechanics, v. 2, pp. 153–225, Academic Press, New York, 1951; cf. these Rev. 12, 764].

H. P. Thielman (Ames, Iowa).

Ščelkačev, V. N. Investigation of the unsteady filtration-flow of an elastic fluid into a circular battery of sinks. Doklady Akad. Nauk SSSR (N.S.) 79, 577–580 (1951). (Russian)

L'auteur examine le stade initial, non stationnaire, d'infiltration d'un liquide élastique à travers les orifices équidistants disposés sur une circonférence. D'après lui, l'abaissement de pression à un point quelconque d'une couche de liquide plane, provenant de la mise en marche de l'orifice d'ordre ν, s'exprime par la formule $\Delta p_\nu = -A\ \mathrm{Ei}\ (-r_\nu^2/4a^2t)$, avec r_ν distance du point examiné à l'orifice (ν), A, a des constantes physiques, Ei fonction intégrale exponentielle. En supposant linéaire l'effet total de n orifices et en utilisant le développement en série de la fonction Ei, on trouve en fin de compte

$$\Delta p = A\left[\log \frac{(4a^2t)^n}{r^{2n}+R^{2n}-2r^nR^n\cos n(\theta-\alpha)} - nC + n\sum_{i=1}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i-1}(r^2+R^2)^{i-2j}r^{2j}R^{2j}}{i(j!)^2(i-2j)!(4a^2t)^i}\right]$$

avec C=constante d'Euler, r=distance du point considéré au centre de circonférence, R=rayon de circonférence, $k=\frac{1}{2}i$ ou $\frac{1}{2}(i-1)$ resp. pour i pair ou impair. En supposant $A=B/n$, on obtient pour $n\to\infty$ la limite

$$\Delta p = B\left[\log\frac{4a^2t}{R^2} - C + \sum_{i=1}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i-1}(r^2+R^2)^{i-2j}r^{2j}R^{2j}}{i(j!)^2(i-2j)!(4a^2t)^i}\right].$$

L'auteur prévoit quelques généralisations de ses formules.

V. A. Kostitsin (Paris).

*Timman, R. **La méthode des caractéristiques et le calcul de la couche limite laminaire en écoulement tridimensionnel.** Actes du Colloque International de Mécanique, Poitiers, 1950. Tome II. Étude sur la mécanique des fluides, pp. 251–259. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 250 (1951).

This is a shorter version of the author's report [Nationaal Luchtvaartlaboratorium, Amsterdam. Report F. 66 (1950);

these Rev. 12, 871] on the same subject. The notation is slightly different; and a number of details and appendices are omitted. *W. R. Sears* (Ithaca, N. Y.).

✱**Comolet, Raymond. Stabilité de l'écoulement laminaire dans la couche limite.** Actes du Colloque International de Mécanique, Poitiers, 1950. Tome III. Étude sur la mécanique des fluides, pp. 231–251. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 251 (1951).

Kurihara, Michinori. A new definition of turbulence based on the theory of probability and the fundamental equations. Rep. Res. Inst. Fluid Eng. Kyushu Univ. 6, no. 2, 6–8 (1950).
Expository paper.

✱**Kampé de Fériet, J. Introduction to the statistical theory of turbulence. Correlation and spectrum.** The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md., 1951. iv+162 pp. $2.25.
The author gives a general review of the subject in the title. He first discusses the (Wienerian) harmonic analysis of an individual function of a single real variable, and then generalizes the discussion to cover vector-valued functions of a point in three-space. He then gives the corresponding treatment of the harmonic analysis of a stationary stochastic process on the group of translations of three-space, which is interpreted as the (spatial) harmonic analysis of a spatially homogeneous turbulent flow at a given time. Under simple regularity conditions, the most general spectral tensor is found for the spatially homogeneous flow of an incompressible fluid, and the vorticity spectral tensor is deduced from this. The difficult problems of the introduction of a varying time, and the role of the Navier-Stokes equations are discussed in some detail. A careful derivation of Loitsianskii's law for the decay of turbulence is given. The book is distinguished throughout by its careful discussion of the physical meaning of the mathematical assumptions.
J. L. Doob (Urbana, Ill.).

✱**Frenkiel, F. N. Introduction to some topics on turbulence.** The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md., 1950. iii+103 pp. $2.15.
The following topics in the theory of turbulence are considered: Diffusion by continuous movements (chaps. I and II; the treatment follows G. I. Taylor [Proc. London Math. Soc. (2) 20, 196–212 (1921)]); an account of the statistical theory of homogeneous isotropic turbulence leading to the von Kármán-Howarth equation (chaps. III and IV; the account following von Kármán and Howarth's original paper [Proc. Roy. Soc. London. Ser. A. 164, 192–215 (1938)] could have been greatly compressed had the notation of isotropic tensors been used); the theory of the measurement of the velocity correlations in a turbulent field by the methods of hot wire anemometry (chaps. V and VI); the relation between the longitudinal and the transverse velocity correlations and the corresponding Fourier representation of the mean square velocity in a given direction (chap. VII). Chapters VIII and IX on "Probability distributions in turbulence" and "Some problems in isotropic turbulence" deal with certain special problems; thus the Loitsiansky invariant is derived in Chapter IX though its meaning in terms of the invariance of the distribution of the largest eddies present is not mentioned. The last chapter X "On locally isotropic turbulence" is a brief introduction to the more recent developments of the subject; an account of Heisenberg's theory and the developments following from it are not mentioned. *S. Chandrasekhar.*

✱**Goldstein, Sydney. Statistical theory of turbulence.** Mimeographed lecture notes prepared by S. I. Pai. The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md., 1950. ii+43 pp. $.80.
Expository lectures on the subject of the title.
J. V. Wehausen (Providence, R. I.).

Goldstein, S. On the law of decay of homogeneous isotropic turbulence and the theories of the equilibrium and similarity spectra. Proc. Cambridge Philos. Soc. 47, 554–574 (1951).
Kolmogoroff's hypotheses that under conditions of local isotropy all the probability distributions governing turbulence depend on the kinematic viscosity ν and the rate of dissipation of energy ϵ are extended by including $d\epsilon/dt$ for describing decaying turbulence. It is shown that on the extended hypotheses the mean square energy of turbulence, $\overline{u^2}$, varies (with a suitably chosen origin of time) as $1/t$; and that quantities and constants which, on Kolmogoroff's original hypotheses, are independent of the Reynolds number R, become functions of R. Arguments are given for supposing that the similarity of the spectrum of $\overline{u^2}$ assumed by Heisenberg [Proc. Roy. Soc. London. Ser. A. 195, 402–406 (1948); these Rev. 11, 63] in his derivation of the $(1/t)$-law must be taken to apply not to the whole of the energy spectrum but only to that part of it in which energy is principally dissipated. It is further suggested that similarity if it exists can be true only asymptotically for large wave numbers. *S. Chandrasekhar* (Williams Bay, Wis.).

✱**Bass, J. La fonction totale de corrélation et son application à l'équation de Karman.** Actes du Colloque International de Mécanique, Poitiers, 1950. Tome III. Étude sur la mécanique des fluides, pp. 267–271. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 251 (1951).
A discussion, similar to that given in the monograph by Agostini and the author [Les théories de la turbulence, same Publ., no. 237 (1950), especially pp. 49–59; these Rev. 11, 751], of the function $R(\xi_1, \xi_2, \xi_3, t) = \sum \overline{u_\alpha u_\alpha'}$, where u_α and u_α' are the components of turbulent velocity at points M and M' and (ξ_1, ξ_2, ξ_3) is the vector MM'.
J. V. Wehausen (Providence, R. I.).

✱**Martinot-Lagarde, André. Remarques sur le spectre de la turbulence.** Actes du Colloque International de Mécanique, Poitiers, 1950. Tome III. Étude sur la mécanique des fluides, pp. 273–280. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 251 (1951).
Discussion of the relation between the mathematically defined correlation functions and spectra of turbulence and the corresponding measured quantities when certain mathematical assumptions concerning the measuring instrument are made. The discussion is restricted to one dimension.
J. V. Wehausen (Providence, R. I.).

✱**Thompson, W. B. Thermal convection in a magnetic field.** Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 196–205. University of Toronto Press, Toronto, 1951. $6.00.
The stability of a horizontal layer of incompressible fluid heated below, in the presence of a magnetic field acting in

the same direction as gravity, is discussed in the manner of Rayleigh [Scientific Papers, v. 6, Cambridge, 1920, pp. 432–446]. It is shown that when both the bounding surfaces are free, instability will set in, in the form of cellular convection, if the temperature gradient $\beta=|dT/dz|$ exceeds the value $\pi^2\mu^2\kappa\sigma H^2/\rho_0\alpha g d^2$ where d is the depth of the layer of fluid considered, ρ_0 is the density, g is the value of gravity, H is the intensity of the magnetic field, μ is the magnetic permeability and α, κ and σ are the coefficients of volume expansion, thermometric conductivity and electrical conductivity (in e.m.u.) respectively. The effect of including viscosity is briefly indicated. *S. Chandrasekhar.*

*Problems of Cosmical Aerodynamics. Proceedings of the Symposium on the Motion of Gaseous Masses of Cosmical Dimensions held at Paris, August 16–19, 1949. Central Air Documents Office, Dayton, Ohio, 1951. iii+v+237 pp. (1 plate).

The papers which are of theoretical interest in this collection are by H. C. van de Hulst [Magneto-hydrodynamic waves, chap. 6, p. 47], J. M. Burgers [Aerodynamical description of elementary expansion phenomena and shock waves, chap. 8, p. 59; The properties of a shock wave in a gas with decreasing density, chap. 14, p. 98; these papers summarize the main results in the author's papers in Nederl. Akad. Wetensch., Proc. **49**, 589–599, 600–607 (1946)], G. C. McVittie [The expansion of an interstellar gas cloud into a vacuum, chap. 9, p. 71; this is essentially the same as the author's paper in Monthly Not. Roy. Astr. Soc. **110**, 224–237 (1950); these Rev. **12**, 642], G. K. Batchelor [Magnetic fields and turbulence in a fluid of high conductivity, chap. 20, p. 149; this is the same as the author's paper in Proc. Roy. Soc. London. Ser. A. **201**, 405–416 (1950); these Rev. **11**, 699], Th. von Kármán [Introductory remarks on turbulence, chap. 19, p. 129; the essentially new points have been published in C. R. Acad. Sci. Paris **226**, 2108–2111 (1948), Proc. Nat. Acad. Sci. U. S. A. **34**, 530–539 (1948); these Rev. **10**, 216, 412] and C. F. von Weizsäcker [Turbulence in interstellar matter (parts 1 and 2), chaps. 22 and 30, pp. 158 and 200; in these chapters a general account is given of the author's papers in Z. Naturforschung **3a**, 524–539 (1948); Z. Physik **124**, 614–627 (1948); these Rev. **10**, 488; **11**, 63]. As indicated, most of the foregoing papers are accounts of papers published by the authors elsewhere. But van de Hulst's paper is an exception. It gives an original account of the propagation of magneto-hydrodynamic waves in a compressible fluid [cf. however, Herlofson, Nature **165**, 1020–1021 (1950)]. By including compressibility, electrical conductivity and viscosity, he is able to give a unified account of damped electromagnetic waves, sound waves, viscosity waves and magneto-hydrodynamic waves. An energy integral valid in all cases is that the sum of the electric and the kinetic energies is equal to the sum of the magnetic energy and the potential energy of compression. If one considers the case of zero viscosity and infinite electrical conductivity and further restricts oneself to velocities of propagation small compared to that of light, then for wave propagation in a direction inclined at an angle φ to that of the magnetic field H we have two modes: When the magnetic field is weak, the two modes correspond to a magneto-hydrodynamic wave (velocity of propagation $V_m=H\cos\varphi/(4\pi\rho)^{\frac{1}{2}}$) and a sound wave (velocity of propagation $V_s=(\gamma p/\rho)^{\frac{1}{2}}$); but as the magnetic field becomes larger, the magneto-hydrodynamic wave tends to a retarded sound wave (velocity of propagation $V_s\cos\varphi$) while the sound wave tends to a modified magneto-hydrodynamic wave (velocity of propagation $V_m/\cos\varphi$).

S. Chandrasekhar (Williams Bay, Wis.).

Ludford, G. S. S. The behavior at infinity of the potential function of a two dimensional subsonic compressible flow. J. Math. Physics **30**, 117–130 (1951).

L'auteur donne la forme la plus générale du développement autour du point à l'infini du potentiel des vitesses φ, pour le mouvement plan, permanent irrotationnel d'un fluide parfait compressible, lorsque la vitesse est inférieure à celle du son. À cet effet l'auteur reprend certaines méthodes utilisées dans l'étude fondamentale qui a été faite sur le sujet par S. Bergman [Trans. Amer. Math. Soc. **62**, 452–498 (1947); ces Rev. **10**, 162]. Dans le plan pseudo-logarithmique, introduit par S. Bergman, la fonction φ et sa conjuguée ψ satisfont, pour un fluide compressible, à une équation différentielle linéaire. L'étude des diverses singularités de φ, qui sont des solutions particulières de cette équation différentielle, permettent d'effectuer le développement de φ et d'en expliciter les premiers coefficients en fonction de la vitesse à l'infini, de la circulation autour de l'obstacle et de l'intensité de la source virtuelle qui peut éventuellement exister dans le champ de la fonction. L'auteur retrouve ainsi en les complétant et les précisant les résultats de Lamb et de Bateman. Enfin l'utilisation du développement de φ permet de calculer les composantes des efforts aérodynamiques subis par l'obstacle et d'étendre au cas du fluide compressible les expressions de la portance et du moment résultant pour un fluide incompressible. *R. Gerber* (Grenoble).

Hansen, A. G., and Martin, M. H. Some geometrical properties of plane flows. Proc. Cambridge Philos. Soc. **47**, 763–776 (1951).

This paper studies the geometrical configuration assumed by the streamline, the Mach lines (when they exist), and by the lines of constant pressure, inclination, speed, and density, at a point in the plane flow of an ideal fluid. The authors prove several theorems describing the possible configurations, and from among these we select two results: (1) In general in the flow of an abstract fluid the directions of the isobar and isocline at a point P lie in the same or different quadrants (the direction of the streamline at P being taken as horizontal) according as the flow is supersonic or subsonic at P, and in the former case they are separated by the Mach directions; (2) in a rotational, isoenergetic flow of a polytropic gas the pencil of lines formed at a point P by the tangents to the streamline and to the lines of constant speed, pressure, and density through P has a cross-ratio equal to the adiabatic exponent γ. Certain results of Lighthill [Proc. Roy. Soc. London. Ser. A. **191**, 323–341 (1947); these Rev. **9**, 391] on the flow geometry at a sonic point are extended here to rotational flows. The general theory is illustrated by three examples. *D. Gilbarg.*

*Cabannes, Henri. Détermination de l'onde de choc attachée, lorsque la vitesse aval à la pointe est subsonique. Actes du Colloque International de Mécanique, Poitiers, 1950. Tome II. Étude sur la mécanique des fluides, pp. 181–196. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 250 (1951).

We consider a plane symmetric obstacle of vertex semi-angle ψ in a uniform supersonic flow with attached shock wave. It is well known that for Mach numbers of the incident flow between two extreme values $M_0(\psi)$ and $M^*(\psi)$ the flow immediately behind the shock at the vertex is

subsonic. Crocco [Atti del Primo Cong. dell'Unione Mat. Ital. 1937, pp. 597–615, Zanichelli, Bologna, 1938] has shown that in this range a power series expansion for the shock line at the vertex is not always possible. This paper obtains and discusses properties of a representation for the shock line valid for all M between $M_0(\psi)$ and $M^*(\psi)$. Let r be the order of the first non-zero derivative of the inclination of the obstacle with respect to its arc length at the vertex. Then the author determines a value $M_r(\psi)$ between $M_0(\psi)$ and $M^*(\psi)$, such that in any flow with $M>M_r(\psi)$ the shock line has a formal power series representation at the vertex, (1) $x=y\cot\beta+a_{r+1}y^{r+1}+\cdots$, whereas for $M_0(\psi)<M<M_r(\psi)$ we have (2) $x=y\cot\beta+Ay^\lambda+\cdots$, in which the expansion is not a power series and $\lambda<r+1$. (In these expressions x is the direction of the incident flow and β the inclination of the shock at the vertex.) The value of λ can be determined explicitly from relations derived from an approximate solution to the flow problem in the neighborhood of the vertex. The method for determining A is obscure. The expansion (2) is valid as well for obstacles terminating in a finite straight segment at the vertex, in which case λ increases with M from unity at $M_0(\psi)$ to infinity at $M^*(\psi)$, and is equal to $r+1$ for $M=M_r(\psi)$ $(r=1, 2, \cdots)$. A consequence of (2) and of the exact determination of λ is to extend and sharpen previous results of Guderley [Tech. Rep. No. **F-TR-2168-ND** (1947), Hqtrs. Air Materiel Command, Wright Field, Dayton, Ohio] on the curvature of the shock waves of non-analytic form considered above. *D. Gilbarg.*

***Carrière, P. Écoulements supersoniques infiniment voisins.** Actes du Colloque International de Mécanique, Poitiers, 1950. Tome II. Étude sur la mécanique des fluides, pp. 197–215. Publ. Sci. Tech. Ministère de l'Air, Paris, no. 250 (1951).

It is assumed that an irrotational flow can be obtained by a small perturbation of a basic one E_0 whose potential is $\Phi_0(x, y)$; x and y being either plane or cylindrical coordinates. The differential equations for the perturbation potential $\epsilon(x, y)$ are written for the following special cases:

$$\Phi=\Phi_0+\beta\epsilon(x, y) \qquad (x, y=\text{plane coordinates}),$$
$$\Phi=\Phi_0+\beta\cos\omega\,\epsilon(x, y) \qquad (x, y, \omega=\text{cylindrical coordinates}),$$

where β is a small quantity defining the magnitude of the perturbation. Next, these equations are replaced by first-order equations for the quantities λ', μ' in characteristic coordinates ξ, η, where λ', μ' are the perturbations of the characteristic variables at any point and ξ, η are coordinates along the characteristics of E_0. This yields a numerical procedure for determination of the perturbed flow pattern. As special cases, flows obtained by perturbation of a parallel stream are considered briefly. After a discussion of typical boundary conditions, the author treats, in detail, the flow about a body of revolution at small inclination. His subsequent examples fall into the category of linearized flows, either conical or about elongated bodies of revolution, and therefore amount to new derivations of known results. [See also the paper by Ferri reviewed below.]

W. R. Sears (Ithaca, N. Y.).

Ferri, Antonio. The linearized characteristics method and its application to practical nonlinear supersonic problems. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2515**, 65 pp. (1951).

It is shown that if the flow field can be represented as a basic flow determined by nonlinear methods and a linearized superposed flow, then the deviation from the basic flow, caused by the change of boundary conditions can be calculated along the same characteristic nets as the basic flow. This method has been applied to two-dimensional rotational flow where the basic flow is potential, to axially symmetrical flow where the basic flow is conical and to slender bodies without symmetry where the basic flow is axi-symmetric. It has also been applied to some three-dimensional wing problems where the two-dimensional flow can be used as basic flow. Examples of such quasi-two-dimensional flows are non-swept wings with certain twist or a spanwise variation of thickness distribution. For a related investigation, see the paper reviewed above. *Y. H. Kuo.*

Melkus, H. Über den abgelösten Verdichtungsstoss. Ing.-Arch. **19**, 208–227 (1951).

In symmetrical plane flow let x_0 be the distance between the intersections B and S of the axis of symmetry with a body and a detached shock of curvature $1/R_s$ at S. To estimate x_0/R_s, approximate the velocity $w_s(x)$ on the axis by its Taylor's series about S up to terms of the second degree in x, inclusive, and solve $w_s(x_0)=0$. The author determines the coefficients by means of the equations for plane flow, the shock conditions, and the assumption that the shock wave is one branch of an hyperbola whose asymptotes are Mach lines of the undisturbed flow. His formula for x_0/R_s as a function of free stream Mach number differs from Dugundji's [J. Aeronaut. Sci. **15**, 699–705 (1948); these Rev. **10**, 494], but no explanation is given for the discrepancy, nor is there any comparison with experimental data. *J. H. Giese* (Havre de Grace, Md.).

***Burgers, Johannes M. Nonuniform propagation of shock waves.** Mimeographed lecture notes prepared by S. I. Pai. The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md., 1951. ii+65 pp. $1.40.

In chapter I, several cases of the propagation of a one-dimensional plane shock in the gravitational field are treated and the methods of integration of the equations of motion are outlined. In Chapter II, the problem is generalized to include the symmetrical cylindrical and spherical waves. The last chapter deals with the motion of a gas in a gravitational field. Specifically, the problem is as follows: An infinite column of gas is bounded at its upper end by a plane and extends indefinitely in the downward direction. Initially, the gas is at rest, the pressure inside varies according to the law governing the distribution of pressure in an atmosphere and the pressure on the upper side of the plane is zero everywhere. If sidewise motion is presented, what is the motion in the vertical direction after the bounding plane is suddenly removed? Both the formal solution and its interpretation are given. *Y. H. Kuo* (Ithaca, N. Y.).

Lighthill, M. J. The energy distribution behind decaying shocks. I. Plane waves. Philos. Mag. (7) **41**, 1101–1128 (1950).

In the theory of plane shock developed by Friedrichs [Comm. Pure Appl. Math. **1**, 211–245 (1948); these Rev. **10**, 638], the disturbance in back of the shock is approximated by a "simple wave", in back of which the fluid is taken to be in its original undisturbed state. The author makes the following accuracy hypothesis: the errors in the Friedrichs theory are comparable with the changes produced in the simple wave by: (i) altering the specific entropy in the simple wave by an amount of order the cube of the maximum shock strength; (ii) altering the velocity distribu-

tion in the simple wave by an amount whose ratio to the velocity of sound is of order the cube of the maximum shock strength. Under this hypothesis, considerable value is claimed for the Friedrichs theory in the large, even though the theory substitutes a constant specific entropy for the variable specific entropy distribution in back of the shock. As a check on the accuracy hypothesis the author considers the shock wave generated by a piston which moves impulsively into a long tube filled with a homogeneous fluid (with perfectly general thermodynamical properties) at rest and then is subsequently retarded in an arbitrary manner till it comes to rest. The increase in energy, kinetic and internal, of the fluid in the simple wave over its internal energy in the undisturbed state is calculated on the "accuracy hypothesis" and compared with the work done by the piston. The difference is assigned to the fluid behind the simple wave and the pressure distribution in the region deduced to a first approximation. The fact that the same pressure distribution is deduced as a result of reflection of the simple wave at the shock, followed by reflection at the piston, affords the author's check on his "accuracy hypothesis". The paper concludes with a treatment of an N-wave. *M. H. Martin* (College Park, Md.).

Oswatitsch, Klaus. Der Kompressibilitätseffekt bei schlanken Rotationskörpern in Unter- und Überschallströmung. Arch. Math. **2** (1949–1950), 401–404 (1951).

Well-known results [e.g. Laitone, J. Aeronaut. Sci. **14**, 631–642 (1947); these Rev. **9**, 392] are derived here by a new and concise procedure. *W. R. Sears.*

Sahliger, K. Ein einfaches Verfahren zur Bestimmung der aerodynamischen Kennwerte von dünnen Profilen. Österreich. Ing.-Arch. **5**, 310–322 (1951).

After a review of the well-known Birnbaum-Glauert thin-airfoil theory, the author shows how numerical calculations can be simplified by expressing the skeleton-line curve by Bessel's interpolation formula. Coefficients useful in this method are tabulated. *W. R. Sears* (Ithaca, N. Y.).

Schultz-Piszachich, W. Beitrag zur formelmässigen Berechnung der Geschwindigkeitsverteilung gewölbter Tragflügelprofile in Unter- und Überschallströmung. Österreich. Ing.-Arch. **5**, 226–240 (1951).

Cambered arifoils are discussed along the following lines. In linearized irrotational sub-(super-)sonic flow, with $+(-)\beta^2=1-M^2$, the velocity potential satisfies $\beta^2\varphi_{xx}+\varphi_{yy}=0$. Let $a'(x)=da/dx=(\varphi_x)_{y=0}-W\cos\alpha$ and $b(x)=(\varphi_y)_{y=0}-W\sin\alpha$. Then

$$\varphi=W(x\cos\alpha+y\sin\alpha)+\sum_{\nu=0}^{\infty}(M^2-1)^{\nu}\left[\frac{d^{2\nu}a}{dx^{2\nu}}\frac{y^{2\nu}}{(2\nu)!}+\frac{d^{2\nu}b}{dx^{2\nu}}\frac{y^{2\nu+1}}{(2\nu+1)!}\right].$$

In supersonic flow $a_\pm'=\mp b_\pm/\beta$, where the subscript $+(-)$ will refer to the upper (lower) surface of an airfoil. In subsonic flow consideration of the potentials of source and vortex distributions leads to

$$\beta\pi a_\pm'(x)=\int_{-1}^{+1}[b_\beta(\xi)\pm f(x)b_\alpha(\xi)/f(\xi)](x-\xi)^{-1}d\xi,$$

where

$$f(x)=(1-x)^{1/2}(1+x)^{-1/2},\quad b_\alpha=\tfrac{1}{2}(b_+ + b_-),\quad b_\beta=\tfrac{1}{2}(b_+ - b_-).$$

Let $a_\pm=\sum_0^\infty \epsilon^\nu a_{\nu\pm}$ and $b_\pm=\sum_0^\infty \epsilon^\nu b_{\nu\pm}$, where ϵ is of the same order as the small angle of attack α. On the airfoil $y_\pm=\epsilon\eta_\pm(x)$, the boundary condition $dy/dx=\varphi_y/\varphi_x$ yields $b_{0\pm}=-W\sin\alpha$, $b_{1\pm}=\eta_\pm'(W\cos\alpha+a_\pm')+\eta_\pm\beta^2 a_{0\pm}'$, etc. To the first order in ϵ and α (linearized theory) $b_\pm=W(y_\pm'-\alpha)$, and $\alpha+b_\alpha/W$ and b_β/W are the slopes of the airfoil's mean line and thickness distribution. In incompressible flow ($M=0$) the zeroth and first order approximations, which involve a_0, b_0, a_1, b_1, lead to various source and vortex distributions, based on airfoil thickness and mean line, previously found by Birnbaum [Z. Angew. Math. Mech. **3**, 290–297 (1923)], Föttinger [Jahrbuch der Schiffbautechnischen Gesellschaft **25**, 295–344 (1924)], and Helmbold and Keune [Luftfahrtforschung **20**, 77–80, 81–96, 152–170, 192–206 (1943); these Rev. **5**, 22, 136]. The author also exhibits source and vortex distributions corresponding to the second order approximation and remarks that approximations of any order in ϵ can be treated similarly by an iteration process.

J. H. Giese (Havre de Grace, Md.).

Moore, Franklin K. Unsteady laminar boundary-layer flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2471**, 33 pp. (1951).

This report deals with the flow in the compressible boundary layer over an insulated plate moving with a time-dependent velocity. It is shown that if the plate moves with a constant acceleration, the flow problem involves one parameter ζ, namely, the ratio of the distance aft of the leading edge to the distance the plate has travelled; for large ζ the flow is of the classical "starting from rest" type and for small ζ the flow is quasi-steady. In the case of arbitrary motion of the plate, the flow is shown to be governed by a group of non-dimensional parameters, depending on the distance aft of the leading edge, the velocity and its time derivatives. The solution for small parameters is again assumed to be quasi-steady and the first order deviation from this flow has been calculated. The cases of the unsteady laminar flow with pressure gradient and the unsteady turbulent boundary layer along flat plate are discussed and similar criterion for quasi-steady state is given.

Y. H. Kuo (Ithaca, N. Y.).

Gilbarg, David. The existence and limit behavior of the one-dimensional shock layer. Amer. J. Math. **73**, 256–274 (1951).

Under conditions concerning the thermodynamical behavior of a viscous fluid which are shown to be equivalent to those on which the reviewer's investigation on shock layers was based [Comm. Pure Appl. Math. **2**, 103–122 (1949); these Rev. **11**, 626] the author proves by an astonishingly simple discussion of signs the existence and uniqueness of a shock layer if heat conductivity λ and viscosity μ (or rather the one viscosity coefficient which matters for the one-dimensional problem) are given as arbitrary functions of the thermodynamical state. A second theorem asserts that with $\lambda\to 0$, $\mu\to 0$ the solution (if proper precaution is taken in fixing the arbitrary additive constant inhering in the coordinate x which appears as parameter) converges toward the well-known discontinuous shock wave in an ideal fluid. Also the separate limiting processes where one of the two coefficients λ, μ tends to zero while the other is kept unaltered are studied and shown to yield the expected results. The paper marks a decisive progress in providing a firm mathematical foundation for an important branch of the theory of the steady flow of fluids of fairly general nature.

H. Weyl (Zurich).

***Harkevič, A. A. Neustanovivšiecya volnovye yavleniya. [Transient Wave Phenomena].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 202 pp.

The basic notions of this work are the plane and the spherical shockwave. Denoting by $\sigma_0(t)$ a function such that $\sigma_0(t)=0$ $(t<0)$, $=1$ $(t\geqq 0)$, these waves are described respectively by potential functions

$$\phi=\sigma_0(t-x/c), \quad \phi=(4\pi r)^{-1}\sigma_0(t-r/c).$$

The author undertakes a detailed study of various radiating systems, and of reflection, refraction and diffraction in simpler cases. The method is usually that of direct composition of elementary waves; the wave-equation and boundary problem techniques are rarely used. There are two chapters, the first purely expository. Chapter II opens with an account of operational calculus and of functions similar to the delta-function. Next considered are the reflection and refraction of plane shock waves, and the plane waves set up by a piston in a closed tube. There follows a very detailed investigation of the field of various line-, plane- and spherical distributions of point sources. There is then an exposition of the author's work on horns and on diffraction by screens [Akad. Nauk SSSR. Žurnal Tehn. Fiz. **19**, 822–827, 828–832, 833–838 (1949); these Rev. **12**, 370; also Doklady Akad. Nauk SSSR (N.S.) **68**, 685–688 (1949); **72**, 45–47 (1950); these Rev. **11**, 755]. Perhaps new, however, is a section on diffraction from a sphere. Other sections relate to radiation resistance, radiation and reception, energy considerations based on the adiabatic gas law, and the transition from transient to stationary waves. The work concludes with a postscript in which the author expounds the advantages of his method over the "spectral method". There are numerous diagrams. The style is throughout extremely clear.

F. V. Atkinson (Ibadan).

Četaev, D. N. On the resistance of a rectangular plate vibrating in a cutout in a plane wall. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 439–444 (1951). (Russian)

Il potenziale di velocità φ si calcola mediante la formula

$$\varphi(P)=\frac{v}{2\pi}\int\int_S \frac{e^{-ikr}}{r}dS$$

essendo S la superficie della piastra e $ve^{i\omega t}$ la sua velocità di vibrazione. L'impedenza di radiazione si calcola poi con

$$Z=\frac{i\sigma\omega}{v}\int\int_S \varphi dS$$

(σ densità del mezzo) e dipende quindi da un integrale quadruplo. Introducendo il raggio vettore ρ sul piano di S con l'origine nella proiezione del punto P sul piano stesso, si esegue facilmente una prima integrazione rispetto a r, ottenendo

$$Z=\sigma v\int\int_S\left[1-\frac{1}{2\pi}\oint_C e^{-ik\rho(s)}\frac{1}{\rho(s)}\left[1-\left(\frac{d\rho}{ds}\right)^2\right]^{\frac{1}{2}}ds\right]dS$$

dove C è il contorno di S e s l'ascissa lungo di esso. L'autore dimostra che in qualunque caso si ha

$$Z=\sigma vS[1-Y(ka)+iX(ka)]$$

essendo a una qualsiasi dimensione lineare di S e X, Y due funzioni che dipendono unicamente dalla forma di S.

Le formule vengono applicate al caso di una piastra rettangolare, arrivando ad integrali semplici, di cui l'integrando contiene l'integralseno e l'integralcoseno. L'autore ha effettuato il calcolo numerico per una piastra quadrata e ne dà i risultati in una tabella. *G. Toraldo di Francia.*

Kaspar'yanc, A. A. On the propagation of sound from a plane pulsating radiator. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 445–450 (1951). (Russian)

Viene studiato il caso non stazionario dell'inizio o della cessazione all'istante t_0 delle vibrazioni armoniche di una piastra piana S, la cui velocità normale è $ve^{i\omega(t+t_0)}$. Il semplice risultato è che per l'avviamento della vibrazione il potenziale di velocità è

$$\varphi(x,y,z;t)=\frac{v}{2\pi}e^{i\omega(t+t_0)}\int\int_{S_e}\frac{e^{-ikr}}{r}dS$$

e per la cessazione della vibrazione

$$\varphi(x,y,z;t)=\frac{v}{2\pi}e^{i\omega(t+t_0)}\int\int_{S_i}\frac{e^{-ikr}}{r}dS$$

essendo S_e e S_i rispettivamente la parte di S esterna ed interna ad una sfera di raggio ct col centro in (x, y, z).

G. Toraldo di Francia (Firenze).

Broer, L. J. F. On the propagation of energy in linear conservative waves. Appl. Sci. Research A. **2**, 329–344 (1951).

This paper shows that for conservative systems, when the stationary phase method applies, the rate of energy propagation in a group of waves is the group velocity and that the conservation of energy governs the choice of an expression for the energy density so that quotient of the average rate of work done by the average energy density is the group velocity. [There are obvious relationships to an overlooked paper of the reviewer [Philos. Mag. (7) **21**, 1033–1056 (1936)] in which (a) the stationary phase method was used for a similar purpose and (b) the group velocity is obtained even for a case where there is dissipation.]

D. G. Bourgin (Urbana, Ill.).

Hirschfelder, J. O., and Curtiss, C. F. The theory of flame propagation. II. J. Phys. Colloid. Chem. **55**, 774–788 (1951).

This is a continuation of previous papers by the same authors [J. Chem. Phys. **17**, 550–555, 1076–1081 (1949)] where they have derived the complete equations describing a reacting gaseous mixture and applied them to the analysis of the microstructure of combustion front. The authors point out quite rightly that there is a basic question, not treated here, about the deviation of the different internal degrees of freedom of the intermediate products from equilibrium at the local kinetic temperature. At present, spectroscopic evidence points to the existence of such deviations. The authors then show that in the absence of flame holder, the flame propagation velocity will be necessarily infinite as claimed by Emmons, Harr and Strong [Report HUX-9, Computation Laboratory of Harvard University, Cambridge, Mass., 1950], and thus the combustion can be considered as unstable. The mathematical problem is then formulated with the authors' flame holder. The authors then discuss the possibility of non-monotone temperature increase across the flame thickness due to the small activation energy for the chain-propagation reactions and the large activation energy for the chain-carrier formation. The effect of pressure on the mechanism of combustion front is next

considered. The authors conclude that generally the pressure simply thickens the front without changing the relative importance of diffusion and chemical kinetics. The second section of this paper discusses the mathematical problem of calculating flame speed. First the character of the solution near the hot boundary where all reaction ceases is analysed to show the multiplicity of linearly independent solutions. The paper concludes with a suggested approximate solution of the differential equations based upon the use of the hot boundary solution as the first step in the iteration process. The possibility of multiple flame speed is again indicated.

H. Tsien (Pasadena, Calif.).

Elasticity, Plasticity

Prager, W. The extremum principles of the mathematical theory of elasticity and their use in stress analysis. University of Washington. Engineering Experiment Station. Bulletin no. 119, 29 pp. (1950).

These Walker-Ames Lectures present the method of the hypercircle [W. Prager and J. L. Synge, Quart. Appl. Math. 5, 241–269 (1947); these Rev. 10, 81] to engineers on a scale of increasing generality. The method is first applied to statically indeterminate structures, in which case the abstract space involved has only a finite number of dimensions. Great attention is given to terminology, and numerical examples are worked out using the orthonormalisation process of M. O. Peach [Bull. Amer. Math. Soc. 50, 556–564 (1944); these Rev. 6, 52]. The method is then applied to the equilibrium of elastic solids, with examples. Bounds for displacement at an interior point are discussed.

J. L. Synge (Dublin).

Sternberg, E., Eubanks, R. A., and Sadowsky, M. A. On the stress-function approaches of Boussinesq and Timpe to the axisymmetric problem of elasticity theory. J. Appl. Phys. 22, 1121–1124 (1951).

The stress function approaches are given in general orthogonal axisymmetric curvilinear coordinates. Boussinesq stress functions are found which are equivalent to Timpe functions.

D. C. Drucker (Providence, R. I.).

Colombo, Giuseppe. Sulla stabilità delle configurazioni di equilibrio di una superficie flessibile ed inestendibile. Rend. Sem. Mat. Univ. Padova 19, 214–230 (1950).

The internal stresses occurring in a flexible, inextensible membrane were studied by E. Beltrami [Opere Matematiche, v. 3, Hoepli, Milan, 1911, pp. 420–464]. E. Laura [Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 346–352, Rome, 1942; these Rev. 8, 381] sought to make more precise the concept of stability of an equilibrium position of a membrane by studying the effect of small disturbances about the given configuration, and showed that if the internal stresses of the given equilibrium configuration have throughout the character of a tension, then the system admits small free oscillations about the equilibrium position. The present author [Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 106, 172–179, 180–183 (1948); these Rev. 10, 748] considered small oscillations about a conical configuration of equilibrium, and was confronted with the need of supposing that the internal stress state was a tension. In the present paper the author observes that there exist equilibrium configurations which seem intuitively stable, but whose internal stresses are not of the character of a tension. Further, an example is given of an equilibrium configuration, whose internal stresses are not of the character of a tension, but still admits small oscillations about the equilibrium position.

J. B. Diaz (College Park, Md.).

Conte, Samuel D. The circular plate with eccentric hole. Quart. Appl. Math. 9, 435–440 (1952).

Starting from the well known equation $D\nabla^2 w+P(x, y)=0$, for the small deflection w of a thin, uniform, homogeneous, and isotropic plate of flexural rigidity D, where $P(x, y)$ is an arbitrary load per unit area acting normal to the plate, the author gives a method for the solution of this equation when the plate is circular with an eccentric hole. Using bi-polar coordinates he gives the solution as an infinite series of biharmonic functions, and indicates how the arbitrary constants involved may be determined under the boundary conditions of clamped or simply supported edges. The particular integrals of the equation are determined when the load function $P(x, y)$ is an analytic function expanded in its Taylor's series arranged in homogeneous powers of degree n in x and y together. When the load function $P(x, y)$ has certain specialised forms, it is shown that a closed form for the solution of the plate with clamped edges may be determined, but if the plate is simply supported the same load functions do not yield a solution in closed form.

R. M. Morris (Cardiff).

Vaĭnberg, D. V. On the computation of composite disks and plates under the action of concentrated forces. Doklady Akad. Nauk SSSR (N.S.) 80, 721–724 (1951). (Russian)

Uflyand, Ya. S. The bending of a sectorial plate with a clamped contour. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 515–518 (1951). (Russian)

The determination of the deflection of a clamped sectorial plate consists in finding a function $u(r, \theta)$ (r and θ polar coordinates in the plane) satisfying inside the sector the partial differential equation

$$\Delta^2 u=q(r, \theta),$$

(where Δ is the Laplacian and q is a given function, the external load on the plate) and the boundary conditions

$$u=0, \quad \frac{\partial u}{\partial \theta}=0, \quad \text{for} \quad \theta=\pm\gamma,$$

$$u=0, \quad \frac{\partial u}{\partial r}=0, \quad \text{for} \quad r=R,$$

where $R>0$ and $0<\gamma<\pi$ are given numbers. In previous papers [G. A. Grinberg and Ya. S. Uflyand, Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 413–434 (1949); Ya. S. Uflyand, Doklady Akad. Nauk SSSR (N.S.) 72, 251–254 (1950); these Rev. 11, 628], a method for the solution of this problem was given, in the form of an integral equation (which may be solved approximately) for the solution. This method did not allow the determination of the bending moment on the rectilinear part of the boundary of the sector, and decreased in accuracy as the angle γ decreased to zero. In the present paper the author presents a method for the solution of the same problem which is free from both these disadvantages. This method consists in seeking a solution for the deflection u in the particular form

$$u(r, \theta)=r\int_0^\infty v(\theta, z)\psi(r, z)dz$$

where

$$\psi(r, z) = \cos\left(z \log \frac{R}{r}\right) + \frac{1}{z} \sin\left(z \log \frac{R}{r}\right)$$

which automatically satisfies the boundary condition $\partial u/\partial r = 0$ for $r = R$. An approximate method of solution is based on the resulting equations satisfied by $v(r, \theta)$. Applications are made to the approximate determination of the bending moment on the rectilinear boundary when there is a single concentrated load at $r = r_0$, $\theta = 0$, and to the exact solution of the problem for the infinite wedge ($R = \infty$).

J. B. Diaz (College Park, Md.).

Klitchieff, J. M. Über die Biegung rechteckiger Platten. Bull. Acad. Serbe Sci. Cl. Sci. Tech. (N.S.) **2**, 69–76 (1951).

The author considers the small deflections of a rectangular plate clamped on all edges. The method is roughly equivalent to one appearing elsewhere [S. Timoshenko, Theory of Plates and Shells, McGraw-Hill, New York, 1940, pp. 222–232]. In the case of simply-supported edges, the Navier solution for a general transverse load is used to get the solution for a general concentrated load. This solution then leads to the one for a general concentrated edge moment. Integration then yields the solution for the simply supported rectangular plate under general distributed edge moments, the solution being expressed in terms of the Fourier coefficients of these edge moments. Introduction of the conditions for clamped edges then yields an infinite set of equations to be solved for the above-mentioned Fourier coefficients. These equations can be solved by successive approximations.

G. E. Hay (Ann Arbor, Mich.).

Klitchieff, J. M. Biegung eines Balkens veränderlichen Querschnitts. Bull. Acad. Serbe Sci. Cl. Sci. Tech. (N.S.) **2**, 77–82 (1951).

The Euler-Bernoulli formula of technical beam theory has been derived for end loading of uniform beams. However, it is customarily used also when the load is distributed along the beam and when the cross-section varies. The author asserts that for the pure bending of a rather special beam with varying cross-section, the Euler-Bernoulli formula agrees with the exact solution to within three per cent. The particular beam considered has a rectangular cross section of uniform width and linearly varying length. This width is much smaller than the length of the cross section, so the problem can be considered as one of generalized plane stress. This generalized plane stress problem has been solved by C. E. Inglis [Trans. Inst. Nav. Arch. **64**, 253–261 (1922)]. Inglis's solution readily verifies the assertion of the author of the present paper. *G. E. Hay.*

Schürch, H. Beitrag zur Statik des Balkens von endlicher Breite (Statik plattenartiger Träger). I, II. Z. Angew. Math. Physik **2**, 26–34, 92–108 (1951).

In the ordinary technical theory of beams, the width and depth of the cross section are small in comparison with the length of the beam. In the present papers the author considers the case when the width of the cross section is comparable with the length of the beam, so the beam actually resembles a plate. The load acts parallel to the short axis of the cross section, and the beam is considered as built up from a number of layers each perpendicular to this short axis. Each layer simulates some elastic property, such as the presence of ribs for reinforcing, or the presence of a core in sandwich type structures.

The classical theory of thin plates is applied to this structure, yielding a fourth order partial differential equation relating the deflection w and the load, and having variable coefficients. A solution of this equation is sought in the form $w = f_0 + yf_1$, where f_0 and f_1 are unknown functions of the coordinate x which varies along the beam, and y varies across the width of the beam. This means that transverse bending of the beam is suppressed, and two simultaneous ordinary differential equations then arise for the functions f_0 and f_1, which equations can be solved analytically or numerically. It is pointed out that the effect of the suppression of transverse bending can be investigated by two means: (1) by the assumption for w of a polynomial containing higher powers of y; (2) by the division of the beam into a number of strips running across the width of the beam, and by the application of ordinary beam theory to these strips. Some applications to rectangular plates are presented. *G. E. Hay* (Ann Arbor, Mich.).

Abramyan, B. L. Torsion and bending of prismatic rods of hollow rectangular section. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1319**, 24 pp. (1951).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 265–276 (1950); these Rev. **12**, 302.

***Weinstein, Alexander. New methods for the estimation of torsional rigidity.** Proc. Symposia Appl. Math. vol. 3, pp. 141–161. McGraw-Hill Book Co., New York, N. Y., 1950. $6.00.

This paper gives a review of some recent investigations on inequalities and estimates for torsional rigidity. Two methods are presented which, while essentially different, have in common the property that they can be applied to other important problems in mathematical physics. The first method [J. B. Diaz and A. Weinstein, J. Math. Physics **26**, 133–136 (1947); Amer. J. Math. **70**, 107–116 (1948); these Rev. **9**, 211, 480] reduces the estimation of torsional rigidity to the estimation of Dirichlet's integral in Neumann's and Dirichlet's problem. The second method [G. Polya and A. Weinstein, Ann. of Math. (2) **52**, 154–163 (1950); these Rev. **12**, 652] makes use of the idea of Schwarz symmetrization [H. A. Schwarz, Gesammelte mathematische Abhandlungen, v. 2, Springer, Berlin, 1890, pp. 327–340] and yields the following isoperimetric inequality: of all multiply connected cross sections with given area and with given joint area of the holes, the ring bounded by two concentric circles has the maximum torsional rigidity.

J. B. Diaz (College Park, Md.).

Conway, H. D., Chow, L., and Morgan, G. W. Analysis of deep beams. J. Appl. Mech. **18**, 163–172 (1951).

The problem considered is essentially the generalized plane stress problem for a rectangular plate with uniform loads acting over portions of two opposite edges. Of course the problem is characterized by the biharmonic equation in rectangular Cartesian coordinates, and the solution could be obtained as an infinite series of the characteristic functions of this equation. Such a solution would entail the laborious expansion in Fourier series of a set of hyperbolic functions. The authors avoid this difficulty by determining two stress functions. One is an infinite series of the above-mentioned characteristic functions, and satisfies all the necessary conditions except that it yields a normal stress on two edges which should be free. This normal stress is approximated by a polynomial, and by the strain-energy method a second stress function is found corresponding to the case when this nor-

mal stress alone acts on the plate. This second stress function is a polynomial, and is not biharmonic. It seems to the reviewer that the method could easily be extended to the case of general loads subject only to the requirement of statical equilibrium. *G. E. Hay* (Ann Arbor, Mich.).

Beth, Richard A., and Wells, Charles P. Finite deflections of a cantilever-strut. J. Appl. Phys. **22**, 742–746 (1951).

The authors consider a cantilever initially straight, acted upon at the free end by an arbitrary force r making an arbitrary angle θ_0 with the direction of the undeflected cantilever. Finite deflections are considered, but the cantilever is thin so the strains are small. The quantities r and θ_0 are regarded as independent variables. The coordinates of the free end, the angle specifying the tangent at the free end, and the potential energy of the cantilever are all expressed in terms of a function of r and θ_0. This function satisfies a nonlinear partial differential equation. Solutions are found in series form. The results are compared with those obtained in a more cumbersome manner from the classical theory of the elastica. The agreement is good. *G. E. Hay.*

Taylor, J. Lockwood. Impact on beams and plates. Bull. Tech. Univ. Istanbul **2**, no. 2, 17–26 (1949). (English. Turkish summary)

In a previous paper [Trans. Inst. Nav. Arch. **88**, 328–336 (1946)] the author considered the impact loading of an infinite beam, taking into account the shear and rotary inertia of the beam. The solution was obtained in the form of an integral. An approximate solution valid for small values of the time was obtained, the solution consisting of two waves travelling in opposite directions. In the present paper the analogous problem for the infinite plate is considered, with analogous results. *G. E. Hay.*

Segedin, C. M. Note on a penny-shaped crack under shear. Proc. Cambridge Philos. Soc. **47**, 396–400 (1951).

An elastic medium with a penny-shaped crack subject to a uniform shearing stress parallel to the plane of the crack is considered. The problem is reduced to finding a harmonic function $\varphi(x, y, z)$ which is even in z and satisfies boundary conditions on the plane $z=0$ of the form: a linear combination of $\partial^2\varphi/\partial z^2$ and $\partial^2\varphi/\partial x^2$ is constant within the circular area and $\partial^2\varphi/\partial x\partial y$ is zero in the same region. The boundary value problem turns out to be the same as that solved by Green for a penny-shaped crack under normal tension [Proc. Cambridge Philos. Soc. **45**, 251–257 (1949); these Rev. **10**, 649]. The corresponding displacements are found. The strain-energy released when an infinite medium subjected to uniform shear develops such a crack is then calculated. *G. H. Handelman* (Pittsburgh, Pa.).

Ziegler, Hans. Stabilitätsprobleme bei geraden Stäben und Wellen. Z. Angew. Math. Physik **2**, 265–289 (1951).

The stability and vibration of rods and shafts subjected to longitudinal compression and torsion are considered. The equations are written down in three systems of coordinates for rods initially straight but twisted. For buckling under pressure and torsion with certain boundary conditions (e.g., one end built in, one end free) the usual method of seeking a displaced equilibrium configuration fails. An energy study shows the torque then to be non-conservative, the energy depends on the displacement path. Vibrations must then be considered, and the corresponding frequency equation is shown to give unstable vibrations with a torque however small. It is stated that damping must modify this result for small torques. Similar conclusions are deduced for the critical speed of a whirling massless shaft with a mass on the free end. With a torque acting, instability occurs for all speeds of rotation. *E. H. Lee* (Providence, R. I.).

Goodier, J. N., and Plass, H. J. Energy theorems and critical load approximations in the general theory of elastic stability. Quart. Appl. Math. **9**, 371–380 (1952).

The buckled state of an elastic system will be said to be not unstable if the potential energy (the strain-energy of buckling deformation minus the work of the critical loads on the buckling displacement) is positive for any type of displacement differing from that of the buckling displacement of the state under consideration. Thus for a pin-ended Euler column the potential energy can be shown to be positive for any type of displacement other than a sinusoidal one. The sinusoidal buckled state is accordingly not unstable. It is proved that if the buckled form of a general elastic system is not unstable in the sense defined above the energy approximation to the critical loads will be too high. Consequently the usual assumption that energy approximations to the critical loads are too high implies the assumption that there is a buckled state that is not unstable.
H. W. March (Madison, Wis.).

Nowiński, Jerzy. On the incorrectness of a theorem stating the identity of center of twist and center of shear. Arch. Méc. Appl., Gdańsk **3**, 53–60 (1951). (Polish. English summary)

C. Weber [Z. Angew. Math. Mech. **6**, 85–97 (1926)] proved a theorem that in a beam of an arbitrary cross-section loaded by a concentrated force and a couple acting in the plane of the cross-section, the center of shear and the center of twist coincide. Grzedzielski and Nowiński [Technika Lotnicza **7**, no. 1, 2–6 (1939)] showed that the theorem applies to a beam twisted by a couple only, and that the position of the center of twist depends on the type and magnitude of loading, contrary to the assumption on which Weber's theorem was proved. The author illustrates the incorrectness of Weber's theorem on an example of a beam, of the shape of a thin-walled tube with axial constraints and rigid end diaphragm. The loads are a force P and a couple M acting in the plane of the end cross-section. The example shows that the position of center of twist and center of shear coincide only when $P=0$. *T. Leser* (Lexington, Ky.).

Szabó, I. Die achsensymmetrisch belastete dicke Kreisplatte auf elastischer Unterlage. Ing.-Arch. **19**, 128–142 (1951).

The author considers a thick circular disk resting on a smooth elastic foundation. The elastic foundation occupies a half-space, and the elastic constants of the cylinder and foundation differ. Cylindrical coordinates r, θ, z are used, the axis of the cylinder being the line $r=0$. That face of the disk which is not in contact with the foundation is subjected to a normal load which is a function of r only, so the problem is a two-dimensional one. By the use of separation of variables, the solutions of the differential equations satisfied by the displacements in both the disk and foundation are found as infinite series involving Bessel functions, the eigenvalues being the zeros of the Bessel function $J_1(\lambda a)$ where a is the radius of the disk and λ is a separation constant. The arbitrary constants in these solutions are sufficient for the satisfaction of all boundary conditions except the one requiring zero normal stress on the curved edge of the disk. The force and couple resultants per unit length of disk edge are then

computed from the partial solution at hand, and from this solution there are subtracted solutions for the cases when these force and couple resultants act alone. A solution of the original problem is thus obtained through an appeal to St. Venant's principle. In the final analysis there arises the necessity of solving an infinite set of linear algebraic equations for an infinite set of constants. The author proves the existence of a solution, and derives approximate values for the higher members of the set of constants. *G. E. Hay.*

Krzywoblocki, M. Z. On the so-called principle of least work method. Österreich. Ing.-Arch. **5**, 81–96 (1951).

In the principle referred to here, expressions are assumed for some unknown functions pertaining to a loaded elastic body. For example, these functions may be stress components or displacements. The assumed expressions involve unknown parameters. The elastic energy is then minimized with respect to these parameters to yield values of the parameters. This method is used extensively in engineering, and yields solutions which are almost always approximate. The author asserts that the method can yield very poor results unless the particular forms chosen for the unknown functions lead to a total elastic energy with a value which is not larger than the value obtained by any other choice. To demonstrate this, the author considers a cantilever the cross section of which is a hollow thin-walled rectangle. The upper and lower surfaces of the beam are reinforced by longitudinal stringers, and in addition there are m equally spaced ribs going around the beam transversely. The beam is thus divided into m portions. The shearing stress in the upper and lower parts of each portion is expressed as a Fourier series with unknown coefficients. Equilibrium considerations are then applied to the component parts of the beam, which leads to expressions for the stresses in these parts in terms of the unknown Fourier coefficients mentioned above. The total elastic energy is then found and minimized with respect to these coefficients, whence the coefficients are found, and so a solution to the problem is determined. This solution gives results entirely different from those obtained from tests. It is stated that this difficulty arises because the assumed Fourier series do not lead to a total elastic energy which is an absolute minimum, that is, if other expressions were assumed instead of these Fourier series, a smaller elastic energy could be obtained; the reviewer does not agree, for it seems that these Fourier series are quite general, and that the difficulty must be due to some other cause. *G. E. Hay* (Ann Arbor, Mich.).

Volkov, S. D. A generalized condition of plasticity. Doklady Akad. Nauk SSSR (N.S.) **79**, 213–216 (1951). (Russian)

It is pointed out that a simple dependence of the critical sheer stress on the normal stresses fails to satisfy experimental results obtained for quasi-isotropic polycrystalline materials. A new general expression for the critical sheer stress is proposed which includes the mean hydrostatic stress. It is shown that for extreme values of the parameters this expression reduces to well-known plasticity conditions. *H. I. Ansoff* (Santa Monica, Calif.).

Mežlumyan, R. A. The boundary conditions in bending and torsion of thin shells beyond the elastic limit. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 537–542 (1950). (Russian)

A variational principle is used to obtain the equations of a thin shell in the region of small plastic deformation. Results obtained in a previous paper by the author [same journal **14**, 253–264 (1950); these Rev. **12**, 373] are used in the derivation. *H. I. Ansoff* (Santa Monica, Calif.).

Ševčenko, K. N The elastic-plastic problem for a heavy half-space with a vertical cylindrical cut-out. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 587–592 (1950). (Russian)

The author considers a half-space with a cylindrical hole extending through it perpendicular to the surface. The surface is assumed free from loading, and stress and strain distributions due to the weight of the half-space are determined. The material is assumed to be isotropic and incompressible. In the plastic range the intensities of stress and strain are taken to be linearly related. *H. I. Ansoff.*

Wu, M. H. Lee. Analysis of plane-plastic-stress problems with axial symmetry in strain-hardening range. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **1021**, 23 pp. (1951).

Issued earlier as Tech. Note Nat. Adv. Comm. Aeronaut., no. **2217** (1950); these Rev. **12**, 879.

ter Haar, D. A phenomenological theory of visco-elastic behaviour. I, II, III. Physica **16**, 719–737, 738–752, 839–850 (1950).

A mathematical treatment of the response of viscoelastic materials is presented. Parts I and II deal with "static" experiments (creep at constant stress and stress-relaxation at constant strain). Part III deals with steady-state response to periodic (sinusoidal) excitation, and with wave-propagation through viscoelastic media. Various methods of approximation are developed and compared. *T. Alfrey.*

Seeger, Alfred, und Kochendörfer, Albert. Theorie der Versetzungen in eindimensionalen Atomreihen. II. Beliebig angeordnete und beschleunigte Versetzungen. Z. Physik **130**, 321–336 (1951).

Im ersten Teil der vorliegenden Arbeit [Z. Physik **127**, 533–550 (1950); diese Rev. **12**, 304] wurde ein eindimensionales Kristallmodell (Atomreihe) betrachtet, auf dessen Glieder ein periodisches Potential (das von den Nachbarreihen herrührt) einwirkt und in der überelastische Atomverschiebungen auftreten. Zwischen den dimensionslosen Variablen, Verschiebung u, Ort x, und Zeit y, erhält man dann die Differentialgleichung

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=\sin u.$$

Die statischen Lösungen lauten

$$u=u_0(x)=\pi+2\operatorname{am}\left(\frac{x}{k}+c_1, k\right) \quad \text{für} \quad 0\leqq k\leqq 1,$$

$$u=u_0(x)=2\arcsin\left(k\operatorname{sn}(x+c_2, k)\right)+\pi \quad \text{für} \quad 1\leqq k^2\leqq\infty.$$

k bedeutet hier $1/k$. An diese statischen Lösungen schliessen sich die der gleichmässig bewegten Versetzungen stetig an. Die erhält man mit Hilfe einer die Schallgeschwindigkeit enthaltenden Lorentztransformation. Näherungsweise wird die zeitabhängige Gleichung mit dem Ansatz

$$u=u_0(x^*)+u_1(x, y)$$

gelöst. Auf diesem Wege werden die Wechselwirkungen von Versetzungen untereinander und mit der Kristalloberfläche behandelt, wobei zusätzliche Kräfte auftreten und ausser-

dem die Bewegung von Versetzungen unter der Wirkung von periodischen Schubkräften besprochen; die bewegen sich überhaupt nicht als starre Gebilde, wie man das bis jetzt angenommen hat. Zum Schluss werden noch einige Bemerkungen bezüglich des zwei- und dreidimensionalen Problems gemacht. *Th. Neugebauer* (Budapest).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Hopkins, H. H. The concept of partial coherence in optics. Proc. Roy. Soc. London. Ser. A. **208**, 263–277 (1951).

L'autore generalizza il concetto di coerenza parziale, introdotto da van Cittert [Physica 1, 201–210 (1934)] e da Zernike [Physica 5, 785–795 (1938)], in modo da tener conto della fase relativa delle due vibrazioni e di una possibile differenza di ampiezza. Se due punti P_1, P_2 sono illuminati da una sorgente estesa e piana Σ, che produce in essi rispettivamente gl'illuminamenti I_1, I_2, si può definire il fattore di coerenza γ_{12} così

$$\gamma_{12}=(I_1I_2)^{-1}\int_\Sigma u_1\bar{u}_2 d\sigma$$

essendo u_1, u_2 rispettivamente le ampiezze complesse prodotte in P_1, P_2 dall'elemento $d\sigma$ della sorgente. La conoscenza di γ_{12} permette di prevedere completamente gli effetti interferenziali delle vibrazioni provenienti da P_1, P_2, qualunque trasformazione ottica si faccia successivamente subire ad esse.

Si trova un teorema analogo a quello di Zernike; γ_{12} risulta eguale all'ampiezza complessa che ha in P_2 la figura di diffrazione centrata in P_1, che competerebbe a un'apertura eguale a Σ. L'autore si occupa poi della propagazione del fattore di coerenza e fa alcune applicazioni.

G. Toraldo di Francia (Firenze).

Friedlander, F. G On the half-plane diffraction problem. Quart. J. Mech. Appl. Math. **4**, 344–357 (1951).

The author's summary of this paper is: "The diffraction of an arbitrary two-dimensional disturbance by a semi-infinite plane screen is considered. It is shown that the solution, at all points of space, can be obtained by a modification of the method developed by Hadamard for Cauchy's problem. The diffraction of the disturbance represented by Hadamard's 'elementary solution' is discussed, and a remarkably simple expression for the diffracted disturbance is found; its Laplace transform is shown to agree with the Green's functions derived from Sommerfeld's two-valued solutions of the wave equation."

The disturbance is specified by a solution u of the equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=f(x, y, t)$$

where, for sound waves, u may be the pressure, and for electromagnetic waves it may be a component of the Hertz vector. *E. T. Copson* (St. Andrews).

Aden, Arthur L., and Kerker, Milton. Scattering of electromagnetic waves from two concentric spheres. J. Appl. Phys. **22**, 1242–1246 (1951).

The paper is concerned with the problem of the scattering of plane monochromatic electromagnetic waves from a sphere covered by a concentric spherical shell. In terms of spherical polar coordinates, the electromagnetic properties of the sphere $r<a$, the shell $a<r<b$ and outer space $r>b$ are specified by different values of the complex propagation constant, complex dielectric factor, complex characteristic velocity and permeability.

The solution, which is naturally rather complicated, is in terms of infinite series whose terms involve spherical harmonics and spherical Bessel functions. It reduces to the well-known Mie-Stratton solution for scattering from a single sphere when b tends to a. The results are said to be important in radar meteorology.

E. T. Copson (St. Andrews).

Hufford, George A. An integral equation approach to the problem of wave propagation over an irregular surface. Quart. Appl. Math. **9**, 391–404 (1952).

The problem of radiowave propagation over an irregular and inhomogeneous surface (like the earth) leads after some suitable simplifications to the wave equation $\nabla^2\psi+k^2\psi=-4\pi\tau$ (k being the propagation constant and τ the distribution of sources) with the boundary condition at the surface S, $\partial\psi/\partial n=-ik\delta\psi$, where δ is a complex constant. At the higher frequencies being used today, radio wave transmission is greatly affected by the irregularities of the surface and it is becoming increasingly important to find some way of estimating the resulting field strengths. The author starts with an integral equation deduced in his master's thesis [G. Hufford, Univ. of Washington, 1948; the deduction is briefly repeated in the present paper]. One might approximate the occurring integral according to Kelvin's principle of stationary phases, that is by summing up only the contributions from the neighbourhoods of points where the phase of the integrand is stationary, these points forming in general a discrete set. This approximation however is not satisfactory in the present case, because in points of the surface S between the source O and the considered point P, the phase, while not stationary, is still but slowly varying. So the surface integral in our equation is to be reduced not to a sum of discrete distributions but to a line integral from O to P. Operating on this principle the author gives a general formula, intended to provide numerical solutions. In the latter part of the paper this formula is applied to some simple cases in order to compare the results to those obtained by direct analytical solution. In the case of a homogeneous plane earth, the result obtained by solving the approximating integral equation with the aid of Laplace transforms agrees almost exactly with that due to Norton [Proc. I.R.E. **25**, 1203–1236 (1937)]. The solution for the case of a spherical earth agrees in many aspects with results obtained by van der Pol and Bremmer [Philos. Mag. (7) **25**, 817–834 (1938)].

H. Bremekamp (Delft).

Ott, Heinrich. Oberflächenwelle und kein Ende. Entgegnung zu Th. Kahan und G. Eckart: "Die Nichtexistenz der Oberflächenwelle in der Dipolstrahlung über ebener Erde" (A.E.U. 5 [1951], 25–32). Arch Elektr. Übertragung **5**, 343–346 (1951).

Cf. Kahan and Eckart, same volume, 25–32 (1951); these Rev. **12**, 776.

Kahan, Théo, und Eckart, Gottfried. Zur Frage der Oberflächenwelle in der Dipolstrahlung über einer ebenen Erde. Entgegnung zu H. Ott: "Oberflächenwelle und kein Ende." Arch. Elektr. Übertragung **5**, 347–348 (1951).

Rydbeck, O. E. H. The theory of magneto ionic triple splitting. Comm. Pure. Appl. Math. **4**, 129–160 (1951).

The author quotes reports of triple splitting from various stations all over the world. He starts from the equations of motion of electrons in a steady magnetic field and an alternating electromagnetic field. The unperturbed wave is expressed by Hankel functions of the first and of the second kind of order $\frac{1}{3}$. Using these expressions the index of refraction is calculated in the neighborhood of the critical angular frequency of the ionospheric medium in question. He then starts to evaluate first and higher order approximations of the third component of the waves. Various numerical results as to refractive index, transmission coefficient, virtual heights of the ionosphere, and polarization angle are shown in graphs. These results are compared with experimental data. *M. J. O. Strutt* (Zurich).

Friedman, Bernard. Propagation in a non-homogeneous atmosphere. Comm. Pure Appl. Math. **4**, 317–350 (1951).

The content of this paper was reviewed earlier as a report [New York University, Mathematics Research Group, Research Rep. No. **EM-28** (1951); these Rev. **13**, 305].

Mirimanov, R. G. On a method of determination of the electromagnetic field inside a closed spherical shell, whose different parts have different dielectric permeability. Doklady Akad. Nauk SSSR (N.S.) **80**, 361–364 (1951). (Russian)

L'involucro sferico viene supposto infinitamente sottile e costituito da un materiale avente la costante dielettrica e la conduttività funzioni assegnate delle coordinate angolari. Le sorgenti sono supposte all'interno. Si utilizzano sviluppi in serie di funzioni sferiche e le condizioni di continuità attraverso all'involucro per la determinazione dei coefficienti. *G. Toraldo di Francia* (Firenze).

Mirimanov, R. G. Radiation resistance of a dipole near an ellipsoid of rotation of high conductivity. Doklady Akad. Nauk SSSR (N.S.) **80**, 189–192 (1951). (Russian)

Il dipolo viene supposto sull'asse dell'ellissoide di rotazione e il problema viene risolto con l'ordinaria tecnica delle coordinate sferoidali e delle relative autofunzioni. La resistenza di radiazione del dipolo risulta sotto forma di un integrale definito contenente tali funzioni. *G. Toraldo di Francia* (Firenze).

Kline, Morris. An asymptotic solution of Maxwell's equations. Comm. Pure Appl. Math. **4**, 225–262 (1951).

This paper was reviewed earlier when it appeared as a report [New York University, Mathematics Research Group, Research Rep. No. **EM-24** (1950); these Rev. **12**, 886].

Levin, M. L. On the geometrical meaning of the conditions for the existence of transverse electric and transverse magnetic fields in curvilinear coordinate systems. Doklady Akad. Nauk SSSR (N.S.) **79**, 589–590 (1951). (Russian)

Let (u, v, w) be a curvilinear coordinate system in space $\mathbf{r}=\mathbf{r}(u, v, w)$, such that there can exist an electromagnetic field in which either the electric or the magnetic vector has no longitudinal w-component. It is then known that the metric coefficients e_u, e_v, e_w, defined by $e_u^2=(\mathbf{r}_u)^2$, etc., must satisfy the conditions $e_w=1$, $\partial(e_u/e_v)/\partial w=0$. It is deduced that the w-surfaces are either concentric spheres or parallel planes. *F. V. Atkinson* (Ibadan).

Bechert, Karl. Ansätze zu einer nichtlinearen Elektrodynamik. Ann. Physik (6) **7**, 369–409 (1950).

It is shown that by making 10 assumptions in classical electrodynamics it is possible to develop a theory of an elementary particle of finite mass, charge, and extension. In particular the solution for a particle at rest is investigated in detail. *C. Kikuchi* (Upton, N. Y.).

Sommerfeld, A., und Bopp, F. Zum Problem der Maxwellschen Spannungen. Ann. Physik (6) **8**, 41–45 (1950).

The Maxwell tensor

$$T_{ik}=\tfrac{1}{2}(H_iB_k+H_kB_i-\delta_{ik}H_jB_j)$$

and a modified tensor

$$T_{ik}=\frac{1}{\mu_0}\left(B_iB_k-\frac{1}{2}B_j^2\delta_{ik}\right)$$

are used to calculate the magnetostatic force density. It is pointed out that the torque computed by using the latter tensor is μ/μ_0 times that of the Maxwell case. *C. Kikuchi* (Upton, N. Y.).

Sommerfeld, A., und Ramberg, E. Das Drehmoment eines permanenten Magneten im Felde eines permeabeln Mediums. Ann. Physik (6) **8**, 46–54 (1950).

The consequences of the theory developed in the preceding review are extended by considering the boundary-value problem of a uniformly magnetized ellipsoid in a homogeneous external field. The case for a prolate ellipsoid is discussed first. The general case of ellipsoid of revolution of arbitrary form is discussed for the case $\mu=\mu_0$. *C. Kikuchi* (Upton, N. Y.).

Fischer, Johannes. Zur Definition der magnetischen Grössen. Ann. Physik (6) **8**, 55–64 (1950).

In the present paper, the author discusses whether the magnetic field intensity H or the magnetic induction B should be used to describe forces and torques on magnetic bodies. *C. Kikuchi* (Upton, N. Y.).

Bondi, H., and Gold, T. On the generation of magnetism by fluid motion. Monthly Not. Roy. Astr. Soc. **110**, 607–611 (1950).

In this paper it is argued that, since in a fluid of infinite electrical conductivity the lines of force are "frozen" into the material, fluid particles at the points of exit and entry of a line of force which possesses an external part must always have been on this line; therefore, if the volume occupied by the fluid is simply connected, there can be no hydrodynamic motion whereby the number of such points of exit and entry can be increased. The authors deduce from this that the magnetic flux integrated over the surface of such a body cannot be increased and that the following theorem is true: "No hydrodynamic motion of a body of perfectly conducting liquid of finite dimensions in an otherwise empty space, possessing originally only arbitrarily weak magnetic fields, can generate an external field of finite moment, provided that the space external to the body is

simply connected (which is always the case if the body is simply connected); but if the external space is multiply connected, an external field can be caused to grow without limit". *S. Chandrasekhar* (Williams Bay, Wis.).

Millar, William. Some general theorems for non-linear systems possessing resistance. Philos. Mag. (7) **42**, 1150–1160 (1951).

It is shown that Kirchhoff's two laws for an electrical network may be replaced by either one of these laws plus a variational principle. An arbitrary functional relationship $f(v, i)=0$ is assumed between the branch voltage v and the branch current i. The "content" of the branch is defined as $\int v\,di$ and the "co-content" as $\int i\,dv$. Making the total content of the network stationary gives Kirchhoff's voltage law. Making the total co-content stationary gives Kirchhoff's current law. If the branches obey the linear relationship of Ohm, content and co-content are both proportional to power dissipated and Maxwell's principle of stationary dissipation results. [A paper by the reviewer [Bull. Amer. Math. Soc. **53**, 963–971 (1947); these Rev. **9**, 285] employs similar ideas and by electromechanical analogy relates the question to the principle of virtual work.] *R. J. Duffin.*

Cherry, Colin. Some general theorems for non-linear systems possessing reactance. Philos. Mag. (7) **42**, 1161–1177 (1951).

This is a companion paper to the paper of W. Millar reviewed above. The magnetic energy of an inductor is $\int i\,d\phi$ where ϕ is the flux linked with the current i. The "co-energy" is defined as $\int \phi\,di$. A network is considered whose branches are nonlinear inductors. It is shown that the total co-energy is stationary with respect to mesh-current variation when the network is driven by current generators. A similar principle is derived for total energy. If a generator is applied to two external terminals, the network is found to behave as a single equivalent inductor. The equations of a general nonlinear L, R, C network are put in Lagrangian form. The content or co-content replaces Rayleigh's dissipation function depending on whether current or voltage coordinates are used. [The result on the equivalent inductor follows also from a paper by the reviewer, Proc. Amer. Math. Soc. **1**, 233–240 (1950); these Rev. **12**, 148.]

R. J. Duffin (Pittsburgh, Pa.).

Quantum Mechanics

Viard, Jeannine. Recherches sur la mécanique ondulatoire non relativiste des systèmes. Collectanea Math. **2**, 175–272 (1949).

The language of J. L. Destouches' Principes fondamentaux de physique théorique [Hermann, Paris, 1942] is used to discuss such ideas, basic to quantum mechanics, as that of representing an observable by an operator in a linear space and in particular of representing momentum by derivatives in configuration space. This lengthy paper culminates in the derivation of expressions for the angular momentum of a rigid rotator known at least since Casimir's thesis [Leiden, 1931]. *A. J. Coleman* (Toronto, Ont.).

Blohincev, D. I. Does there always exist the dualism between waves and particles? Uspehi Fiz. Nauk **44**, 104–109 (1951). (Russian)

The author shows by means of examples that it is possible under certain circumstances to have pairs of interacting wave fields which, on the basis of the laws of quantum theory, fail to display the usual particle properties. Whether such fields exist in nature is an open question.

N. Rosen (Chapel Hill, N. C.).

Vrkljan, V. S. Über die Beziehungen zwischen den drei Diracschen Matrizen. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **6**, 49–56 (1951). (German. Serbo-Croatian summary)

The author discusses a set of 7 new relations for the Dirac matrices α, β, and γ. *I. E. Segal* (Princeton, N. J.).

Rose, M. E. A note on Dirac central field wave functions. Physical Rev. (2) **82**, 389–391 (1951).

Methods of solution of the Dirac equation for a particle in a central field are given with particular reference to the field of a nucleus of non-zero extension. An iteration method provides an expansion near the origin. A concise method of normalization is given for wave-functions of bound states. Phase shifts are discussed for the continuum.

C. Strachan (Aberdeen).

Rose, M. E., and Newton, R. R. Properties of Dirac wave functions in a central field. Physical Rev. (2) **82**, 470–477 (1951).

Solutions of Dirac's equation for a particle in a central field of potential energy $V(r)$ are discussed. A condition, essential for much of the argument, which the authors impose is $\lim_{r\to 0}(F_kG_k - F_kG_l)=0$, where F_k/r, G_k/r are the usual radial factors in the wave function for the state labelled by k. They conclude that no eigensolutions in the discrete or continuous spectrum are admissible if $|\lim_{r\to 0}(rV)|>1$. For $|\lim_{r\to 0}(rV)|\leqq 1$ the behaviour of solutions near $r=0$ is described. Conditions at infinity determine when bound states and/or a continuum occur. The numbers of nodes of F_k and G_k and the displacement of these nodes with changing energy are discussed. Similar and contrasted results are derived for the Schrödinger non-relativistic equation. *C. Strachan* (Aberdeen).

Ludwig, Günther. Die erzwungenen Schwingungen des harmonischen Oszillators nach der Quantentheorie. Z. Physik **130**, 468–476 (1951).

Exact transition probabilities are found for a harmonic oscillator under the influence of an arbitrary time-dependent external force. Dirac's approximation procedure gives the general solution of the corresponding time-dependent Schrödinger equation as a series in powers of the coupling parameter. *T. E. Hull* (Vancouver, B. C.).

Humblet, J. Perturbation des niveaux virtuels. Bull. Soc. Roy. Sci. Liège **20**, 323–326 (1951).

A one-dimensional wave equation with a perturbation of the (Siegert) boundary condition is considered. A formula for the perturbation of the energy levels is derived and discussed; it holds for both real and virtual levels.

T. E. Hull (Vancouver, B. C.).

Peaslee, D. C. Note on the compound Dirac equation. Physical Rev. (2) **84**, 373–374 (1951).

Mrowka, B. Zur Darstellung der Quantenmechanik. I. Unrelativistische Wellengleichungen. Z. Physik **130**, 164–173 (1951).

A method of deriving the Schroedinger equation given by E. Madelung [Die mathematischen Hilfsmittel des Phys-

ikers, 4th ed., Springer, Berlin, 1950; these Rev. 12, 484] is generalized to the case of more complicated wave equations. The method is based on three axioms: (1) the uncertainty, or complementarity, principle; (2) the occurrence of interference phenomena in material systems; and (3) the correspondence principle. From these axioms, together with a number of assumptions, stated or implied, the author derives the non-relativistic wave equation for a spinless particle in an electromagnetic field and the non-relativistic Pauli equation for a particle with spin. *N. Rosen.*

Höhler, G. Zur Theorie der verallgemeinerten Wellengleichung. Ann. Physik (6) 9, 77–90 (1951).

Verf. behandelt lineare Verallgemeinerungen der Wellengleichung innerhalb der klassischen Theorie. Zunächst werden die lorentzinvarianten und zeitsymmetrischen Lösungen der homogenen Mesongleichung bestimmt, deren Fundamentalsystem Besselsche und Neumannsche Funktionen enthält. Man hat quellenfreie und allgemeinere Lösungen zu unterscheiden, wie dies auch aus Untersuchungen von J. Schwinger hervorgeht [Physical Rev. (2) 75, 651–679 (1949), appendix; diese Rev. 10, 663]. Innerhalb der Lichtkegel können die Lösungen der homogenen Mesongleichung noch vom Vorzeichen abhängen, wenn man die Forderung der Zeitsymmetrie fallen lässt. Weiterhin werden zwei Methoden zur Gewinnung zeitsymmetrischer Greenscher Funktionen der Mesongleichung entwickelt. Die zweite dieser Methoden ergibt sich aus der Berechnung der halben Summe der retardierten und avancierten Lösung der Gleichung für die Greensche Funktion der Mesongleichung und führt über die Ergebnisse von Schwinger hinaus. Die Überlagerung symmetrischer Greenscher Funktionen der Mesongleichung beleuchtet die Abgrenzungen von Meson- und Maxwellfeldern. Mesonfelder allein führen nicht auf Coulombfelder. Nach Diskussion der retardierten Greenschen Funktionen der Mesongleichung und der Zusatzterme, welche im Falle nichtzeitsymmetrischer Wellengleichungen mit retardierten Greenschen Funktionen auftreten, werden die in der Literatur vorliegenden Beispiele in den in dieser Arbeit entwickelten Rahmen eingefügt. *M. Pinl* (Dacca).

Höhler, G. Ein Beispiel zur klassischen Feldmechanik. Ann. Physik (6) 9, 91–96 (1951).

W. Heisenberg's "Elementarlänge" führt, dividiert durch die Lichtgeschwindigkeit, auf eine "Elementarzeit" [cf. W. Heisenberg, Ann. Physik (5) 32(424), 20–33 (1938)]. F. Möglich und R. Rompe versuchten entsprechend zeitliche Ableitungen in den Bewegungsgleichungen eines Elektrons durch Differenzenquotienten zu ersetzen, deren Schrittweite durch diese Elementarzeit gegeben ist. Verf. gelangt zur gleichen Bewegungsgleichung wie Möglich und Rompe im Rahmen der Boppschen Theorie des Elektrons, ohne die üblichen Raum-Zeitvorstellungen zu verlassen. Wenn eine Bewegungsgleichung im Rahmen der Boppschen Theorie durch eine Differenzengleichung dargestellt werden soll, erweist sich deren Intervall notwendig gleich der Elementarzeit $\tau=2e^2/3mc^3\approx 6\times 10^{-24}$ sec. *M. Pinl* (Dacca).

Fedorov, F. I. On minimal polynomials of matrices of relativistic wave equations. Doklady Akad. Nauk SSSR (N.S.) 79, 787–790 (1951). (Russian)

The author considers the problem of finding physical requirements which limit the generality of possible relativistic wave equations of the form

$$(\gamma^k(\partial/\partial x_k)+im)\psi=0.$$

He proposes in particular the requirement that there should not exist a solution of the equation for which the energy-density is everywhere zero. Using the algebraic analysis of Gel'fand and Yaglom [Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 18, 703–733 (1948); these Rev. 10, 583] he finds the following necessary condition for non-existence of such solutions: the minimal polynomial equation satisfied by each of the matrices γ^k must be of the form

$$\gamma^n(\gamma^2-\lambda_1^2)(\gamma^2-\lambda_2^2)\cdots(\gamma^2-\lambda_s^2)=0,$$

where n is a non-negative integer and the $\lambda_1, \cdots, \lambda_s$ are unequal positive real numbers. In particular, this condition excludes a theory of nuclear forces proposed by Bhabha [Physical Rev. (2) 77, 665–668 (1950); these Rev. 11, 764] in which the minimal polynomial would have a repeated non-zero root. *F. J. Dyson* (Ithaca, N. Y.).

Cook, J. M. The mathematics of second quantization. Proc. Nat. Acad. Sci. U. S. A. 37, 417–420 (1951).

This note summarizes a longer paper submitted for publication elsewhere. The author has set up a mathematically precise and rigorous formulation of the theory of a linear quantized field, avoiding the use of singular functions. The formalism is equivalent to the usual one, only it is more carefully constructed, so that every operator is a well-defined Hilbert space operator and every equation has an unambiguous meaning. Fields obeying either Fermi or Bose statistics are included. There is no discussion of the non-linear interactions, the understanding of which still presents a major challenge to mathematicians. *F. J. Dyson.*

Feynman, Richard P. An operator calculus having applications in quantum electrodynamics. Physical Rev. (2) 84, 108–128 (1951).

Mathematicians and physicists have always been accustomed to represent the operation of multiplication by simple juxtaposition of the multiplicands. When non-commuting operators were introduced, it seemed natural to adopt the convention that the product AB should represent the operator obtained by operating first with B and afterwards with A. Thus the order of operation of the factors became associated with the order in which the symbols were written on the paper. Later, when non-commutative algebra was axiomatized and made into an abstract discipline, this conventional association of operating order with written order became so firmly established that few people ever paused to question its utility.

In this paper an alternative notation is proposed. Each operator carries an ordering parameter S which is a real variable. The product $A(S)B(S')$ is defined to mean AB if $S>S'$, BA if $S'>S$. When $S=S'$ the product is in general undefined. The new notation has the great advantage over the conventional one, that it makes all the rules of elementary commutative algebra valid for non-commuting operators. As a result of this, the author shows how it is possible also to construct functions of two or more non-commuting quantities in an unambiguous way, so that the ordinary rules of differentiation and integration of such functions remain valid. A very general type of function definable in this way is the following. Let $F[M(S), N(S)]$ be any functional of two functions M, N of the same real variable S, representable by some kind of Fourier integral in function space

$$(1)\quad F[M,N]=\int\int \exp\left[i\int(\mu(S)M(S)+\nu(S)N(S))dS\right]\times\Phi[\mu(S),\nu(S)]d\mu d\nu,$$

where Φ is a functional of μ, ν which are variable functions of S. Then a corresponding function $F(P(S), Q(S))$ of the non-commuting operators P, Q is defined by

$$(2)\quad F(P,Q)=\int\int \exp\left[i\int(\mu(S)P(S)+\nu(S)Q(S))dS\right]\times\Phi[\mu(S),\nu(S)]d\mu d\nu.$$

The exponential function in (2) may be defined by the usual power-series expansion when this is convergent. More generally one may define

$$(3)\quad R=\exp\left[\int_0^1 U(S)dS\right],$$

where $U(S)$ is any operator function of S, as the value $G(1)$ of the solution $G(S)$ of the differential equation

$$(4)\quad dG(S)/dS=U(S)G(S)$$

starting from the initial value $G(0)=I$.

The main part of the paper outlines the applications of the new notation to numerous calculations in quantum mechanics. The notation is particularly well suited to such applications, because many problems in quantum mechanics are reducible to a disentangling of mixed exponentials of non-commuting quantities. In quantum electrodynamics the notation leads to a very marked simplification in the calculations and to an increased understanding of the theory. The results of these calculations, so far as they are of interest to physicists, have mostly been already published by the author [Physical Rev. (2) **80**, 440–457 (1950); these Rev. **12**, 889, and earlier papers there quoted]. A treatment of the Dirac equation by using the ordering parameter as a formal fifth coordinate is here published for the first time. Similar treatments have meanwhile been independently proposed by Y. Nambu [Progress Theoret. Physics **5**, 82–94 (1950); these Rev. **11**, 766] and J. Schwinger [Physical Rev. (2) **82**, 664–679 (1951); these Rev. **12**, 889].

F. J. Dyson (Ithaca, N. Y.).

Visconti, A. Contributions à l'étude de quelques points de la théorie de R. P. Feynman. J. Phys. Radium (8) **12**, 726–734 (1951).

Feynman's method of solving the Schrödinger equation with the aid of a Green's function [Physical Rev. (2) **76**, 749–759 (1949)] is linked up with a "hypothesis of heredity", proposed by V. Volterra [Leçons sur les fonctions de lignes, Gauthier-Villars, Paris, 1913]. According to this hypothesis, the difference between the perturbed evolution operator of a system $U(t, t_0)$ and the unperturbed evolution operator $U_0(t-t_0)$ should depend upon all the values of $U(\tau, t_0)$ where $t_0<\tau<t$. Then

$$U(t,t_0)=U_0(t-t_0)+\int_{t_0}^{t}F(t,\tau)U(\tau,t_0)d\tau$$

where $F(t, \tau)$ represents the "hereditary influence". The conditions which $F(t, \tau)$ must fulfill are discussed and some of Feynman's formulas are derived in a more explicit way. In a second part of the paper it is proposed to interpret Feynman's integral equation as representing diffusion of "effective" and "virtual" de Broglie waves. "Virtual waves" are, by definition, those waves which appear in the calculations only, e.g., the incident wave is "effective" before the perturbation sets in, but "virtual" after that. Use is made of this interpretation in developing an approximation procedure.

E. Gora (Providence, R. I.).

Steinwedel, Helmut. Zum Formalismus linearer Feldtheorien. II. Z. Naturforschung **6a**, 519–522 (1951).

The methods of a previous paper [same Z. **6a**, 123–133 (1951); these Rev. **12**, 888] are applied directly to the Lagrangian. Thus the usual Lagrangian density $L(\varphi, \varphi^*, \cdots; \eta)$ where η depends on the rest mass of the particle type considered, is replaced by $\mathfrak{L}=\int_0^\infty L(\varphi, \varphi^*, \cdots; \eta)\rho(\eta)d\eta$ where $\rho(\eta)$ is the weight function previously introduced. Commutation rules for bosons are obtained and conditions, of the form $\int_0^\infty \eta^m\rho(\eta)d\eta=0$ for $m=0, 1, \cdots, n$, eliminate divergences. An example shows that the formalism will not apply to fermions.

A. J. Coleman (Toronto, Ont.).

Anderson, James L., and Bergmann, Peter G. Constraints in covariant field theories. Physical Rev. (2) **83**, 1018–1025 (1951).

This is the fourth paper by P. G. Bergmann and his coworkers connected with Bergmann's program to bring a general covariant field theory into a canonical form [Physical Rev. (2) **75**, 680–685 (1949); **80**, 81–88 (1950); Rev. Modern Physics **21**, 480–487 (1949); these Rev. **10**, 408; **12**, 292; **11**, 299]. In lagrangian formalism the lagrangian density as a function of the field variables and their first derivatives is postulated and from the invariance properties of the theory a number of identities (generalized Bianchi identities) are derived. In canonical formalism the momentum densities, introduced in customary manner, must satisfy a set of algebraic relations, called primary constraints. Since not all solutions of the canonical field equations satisfy primary constraints and, further, it is not sufficient to satisfy the primary constraints only on one initial hypersurface, the authors establish further additional conditions between canonical variables (secondary constraints) which guarantee that canonical field equations can be solved and the primary constraints remain satisfied everywhere. The number of such constraints is finite and they form a function group; the proof is based on the possibility of constructing a generating function for an infinitesimal canonical transformation. The authors give applications of their results to electromagnetic theory and to coordinate-covariant theories.

M. Brdička (Prague).

Sollfrey, W., and Goertzel, G. Some quantum-mechanical divergences of a simple field. Physical Rev. (2) **83**, 1038–1044 (1951).

The classical and quantum-mechanical properties of a finite or infinite vibrating string coupled to a harmonic oscillator are studied. For the finite string the Hamiltonian is ordered so that the zero-point energy of the lowest state of the coupled system is removed. This leads to a finite energy for an initial state defined, in terms of the uncoupled system, with the oscillator in a definite excited state and no quanta present in the string. This is said not to be true when the zero-point energy of the uncoupled system is removed. The rate of change of the lowest state, defined in terms of the uncoupled system, is infinite. A possible deeper discussion of this difficulty is hinted at but not developed. Additional divergences arise for the infinite string. The validity of the application of perturbation theory to these problems is discussed.

C. Strachan (Aberdeen).

Markov, M. A. On the differences between particles and antiparticles obeying Dirac's equation. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 761–769 (1951). (Russian)

The discussion is organized under the headings. 1) Neutral fields: (a) meson fields. (b) neutrino fields, (c) differences

between neutrino and antineutrino. 2) Charged fields: (a) meson fields, (b) β-forces. The Dirac equation for a nucleon in interaction with one of the above fields is contrasted with that for the antinucleon. Particular attention is devoted to the sign of the specific charge for the five standard transformation types of interaction in order to pick out the "Majorana theories", that is, theories "without antiparticles". The following is typical of the type of result obtained by rather obvious considerations: The specific charge for neutral scalar or pseudovector interaction with antinucleons has the same sign as for that with nucleons, whereas it has the opposite sign in the case of pseudoscalar, vector or tensor interaction. *A. J. Coleman* (Toronto, Ont.).

Husimi, Kodi, and Nishiyama, Toshiyuki. An algebraic theory of the density matrix. I. Progress Theoret. Physics 5, 909–919 (1950).

In this and the paper reviewed immediately below, an attempt is made to combine the theory of the Dirac density matrix as generalized by Husimi [Proc. Phys.-Math. Soc. Japan (3) 22, 264–314 (1940)] with a study of the algebra of the creation and annihilation operators of second quantization to provide a method of discussing n-particle problems which is simpler than previous group theoretical methods. Part I assembles those parts of Husimi's work, referred to above, which are relevant to the present objective. Some formulas involving the partition function of a system of bosons or fermions are obtained, together with an expression for the reduced density matrix belonging to magnetic quantum number M_s for n electrons. Further, a new derivation of the Hartree-Fock equations is given in a form applicable to a configuration in which m orbitals are doubly, and $n-2m$ are singly, occupied. *A. J. Coleman.*

Nishiyama, Toshiyuki. An algebraic theory of the density matrix. II. Progress Theoret. Physics 6, 1–16 (1951).

The program announced in Part I [see the preceding review] is advanced by a study of the structure of the algebra generated by operators in quantum space of the form $C^*(\lambda)C(\lambda)$ where $C(\lambda)=\prod_{i=1}^{n}a_{r_i\lambda_i}$, with $a_{r_i\lambda_i}$ the annihilation operator for the r_i-orbital λ_i-intrinsic state. The intrinsic state λ_i is capable of k (finite) values, for example the two values of spin for an electron or the four spin-charge states of a nucleon. For a particular orbital configuration, the author introduces the notion of an elementary algebra, of dimension 2^k, corresponding to all possible occupations of the intrinsic states. Simple sub-ideals correspond to particular values of the total spin value in the case of electrons, or to Wigner supermultiplets in the case of nucleons. The well-known forms for Bartlett, Heisenberg and Majorana forces appear. Three of the six appendices illustrate the fact that the methods of the paper may be applied to the study of the representations and characters of the permutation group. The author remarks that the paper provides "an interesting way of recognizing old things from a new point-of-view although there are no fundamentally new results." [Reviewer's comment: Due to the condensed style, difficulties of language and lack of clarity in definitions, the reviewer found these papers rather obscure and so was unable to decide whether or not they represent a significant simplification of group theoretical methods. He suspects that they do and would hope that a fuller and more careful presentation of these methods, together with illustrative examples, will appear.] *A. J. Coleman.*

Heisenberg, W. On the mathematical frame of the theory of elementary particles. Comm. Pure Appl. Math. 4, 15–22 (1951).

This article summarizes two previously published papers of the author [Z. Naturforschung **5a**, 251–259, 367–373 (1950); these Rev. **12**, 573]. He proposes a general scheme for a new theory of elementary particles, without making a model specific enough to be worked out in detail.

F. J. Dyson (Ithaca, N. Y.).

Burton, W. K. Lagrangian S-matrix. Physical Rev. (2) 84, 158 (1951).

Klein, Abraham. The coupling of a Dirac field to a Kemmer field. Physical Rev. (2) 82, 639–646 (1951).

The object of the present investigation is to present an approach to the problem of coupling a nucleon field to a Kemmer field in a manner general enough to free completely the theory of any dependence on the usual wave formulation. The author reaches this result by the use of little more than the algebra of the Dirac matrices employing a representation of the Kemmer matrices which contains each irreducible representation just once. He is able to treat spin-zero and spin-one theories uniformly. As a single illustration of the utility of the developed formulations the author calculates the nuclear forces to lowest non-vanishing order. Several cases of interest are considered: pseudoscalar coupling (in connection with the static nuclear potential energy or the Møller interaction of two nucleons), scalar coupling, pseudovector coupling, vector coupling. *M. Pinl.*

Wu, Shi-Shu. A remark on the conventional perturbation theory. Physical Rev. (2) 83, 730–734 (1951).

The author presents a detailed verification of the fact that the results of the Feynman covariant quantum electrodynamics [Physical Rev. (2) 76, 769–789 (1949); these Rev. 11, 765] can be deduced from the old-fashioned Heisenberg-Pauli electrodynamics using standard perturbation theory. On general grounds it is clear that this must be possible, since the Feynman theory is only a reformulation of the old theory. *F. J. Dyson* (Ithaca, N. Y.).

Tyablikov, S. V. An adiabatic form of perturbation theory in the problem of the interaction of a particle with a quantum field. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 21, 377–388 (1951). (Russian)

Except for the last page, this is a summary of a previous paper which developed a form of perturbation theory appropriate to the study of particles in a field when the kinetic energy of the field is small [Bogolyubov Ukrain. Mat. Žurnal 2, no. 2, 3–24 (1950); these Rev. 13, 194]. The method is here applied to an electron in an ionic crystal giving a value for the effective mass which is about 6% less than that given by Landau and Pekar [same Žurnal 18, 419–423 (1948)]. Finally, the author takes issue with recent arguments of Fröhlich, Pelzer and Zienau [Philos. Mag. (7) 41, 221–242 (1950)] against the possibility of "self-trapping" by the electron. He concludes that though they make a valid criticism of Pekar's formulation of the theory, their arguments do not exclude the possibility of self-trapping.

A. J. Coleman (Toronto, Ont.).

Tyablikov, S. V. The excited state of particles in a field. Doklady Akad. Nauk SSSR (N.S.) 81, 31–33 (1951). (Russian)

Results, without proof, are reported for the adiabatic case [see the paper reviewed above] of a spinless particle inter-

acting with a scalar field when the Hamiltonian admits translations and rotations about an axis. A second approximation is given involving new terms which may be interpreted, according to the author, as arising from an effective mass and from a moment of inertia about the axis.

A. J. Coleman (Toronto, Ont.).

Tyablikov, S. V. On the theory of the interaction of a particle with a quantum field. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 16–24 (1951). (Russian)

Let H be the Hamiltonian describing a nonrelativistic charged particle interacting with a quantized Maxwell field. The coordinates describing the state of the system are the occupation numbers $N_f=a_f^*a_f$ of the field-oscillators and the position q of the particle. By a suitable contact transformation these coordinates are replaced by new variables in terms of which the Hamiltonian separates into a part depending only on the particle and a part depending only on the field. In this way, as is well-known, the effects of the vacuum fluctuations of the field on the behaviour of the particle may conveniently be calculated. The effects are, a change in the effective mass of the particle, and a shift in the discrete energy levels of the particle when bound in an atom (Lamb shift). In this paper the electromagnetic mass and level-shifts are calculated, not only for the case of an unexcited radiation field (all $N_f=0$) but also when the field oscillators are excited to a state of thermal equilibrium with temperature T. The temperature-dependent effects are found to be very small under any realizable conditions.

F. J. Dyson (Ithaca, N. Y.).

Galanin, A. D. Radiative corrections in quantum electrodynamics. Doklady Akad. Nauk SSSR (N.S.) **79**, 229–232 (1951). (Russian)

A method is developed which does not require use of the interaction representation, and makes it possible to solve the wave equations for electron and photon fields in interaction: $L\Psi=A\Psi$ and $p^2A=4\pi e^2I$, by the method of successive approximation. Ψ and A are the quantum operators of the electron and of the photon field, and I is the current-density operator. First Ψ is eliminated from the equation for the photon field; this leads to a power series expansion for A. The wave equation for the electron field is then replaced by the integral equation $\Psi=\Psi_0+L^{-1}A\Psi$ $(L\Psi_0=0)$ which can be solved by iteration. Feynman's rules for writing down matrix elements in quantum electrodynamics [Physical Review. (2) **76**, 749–759, 769–789 (1949); these Rev. **11**, 765] are then derived by considering instead of the typical diagrams the corresponding terms in the series expansions.

E. Gora (Providence, R. I.).

Gell-Mann, Murray, and Low, Francis. Bound states in quantum field theory. Physical Rev. (2) **84**, 350–354 (1951).

The Bethe-Salpeter equation [H. A. Bethe and E. E. Salpeter, Physical Rev. (2) **82**, 309–310 (1951)] is derived on the basis of conventional field theory. A definition of the two-body kernel in terms of "true vacuum" expectation values is given. Comparing power series expansions this is shown to be the same as that defined by Feynman [ibid. (2) **76**, 769–789 (1949); these Rev. **11**, 765]. The kernel is written as a sum of wave functions. A certain limiting procedure makes possible the separation of the wave function corresponding to a definite bound state. This combined with the integral equation for the kernel yields the Bethe-Salpeter equation for the bound state wave function.

K. M. Case (Ann Arbor, Mich.).

Neuman, Maurice. A note on the U operator. Physical Rev. (2) **83**, 671–672 (1951).

The author defines the "Heisenberg collision operator" by

$$S^r=\bar{U}^s(t)U^r(t), \quad S^s=\bar{U}^r(t)U^s(t)$$

and the "Heisenberg reaction operator" by

$$\bar{U}^s(t)U^r(t)=U^s(\infty), \quad \bar{U}^r(t)U^s(t)=\bar{U}^s(\infty),$$

where the dash denotes hermitian adjoint. It is usually assumed that the operator $U(t)$ transforming the state vector from the interaction to the Heisenberg representation satisfies the relation (*) $\bar{U}(t)=U^{-1}(t)$. One then infers from (*) that (**) $\bar{S}=S^{-1}$. Here $S=\lim_{t\to\infty}U(t)$. The author shows in this note that (**) is implied by the formal structure of the theory, statement (*) however is not.

M. Pinl (Dacca).

Gregory, Christopher. On the formal expansion of operator functions in terms of a set of basic operators. Proc. Nat. Acad. Sci. U. S. A. **37**, 706–709 (1951).

A formal expansion of a non-local field operator in terms of orthogonal functions of the space-time operators x^μ and the displacement operators p_μ is obtained. It is shown that the expansion coefficients are independent of the representation chosen for the latter operators. Questions of legitimacy and interpretation are not discussed.

K. M. Case.

Franz, W., und Tewordt, L. Die Multipole des Mesonenfeldes. Z. Physik **130**, 457–467 (1951).

The general solution of the Kemmer equation with static spherically symmetric potential, $\beta^r\partial_r\psi+k\psi=0$, where $\partial_4=\partial/\partial x^4+(e/\hbar c)\Phi(r)$, $\partial_\alpha=\partial/\partial x^\alpha$ $(\alpha=1, 2, 3)$, $r^2=x_\alpha x^\alpha$, is studied in the case of scalar (β^r: 5-dimensional) and vector mesons (β^r: 10-dimensional). The solution is obtained as a sum of multipole solutions which are eigenvectors of the total angular momentum and its third component. The multipole solutions are exhibited explicitly in terms of scalar functions of r which satisfy linear second order differential equations involving Φ. Great simplicity, compared with the usual treatments, is obtained by a consistent use of the familiar vector operator ∇.

A. J. Coleman.

Proca, A. Transmutation des particules fondamentales. Changement de spin. J. Phys. Radium (8) **12**, 123–130 (1951).

Fundamental particles are assumed to be different states of a single particle. Spin is taken as basic since its proper values are known. Integral values of spin are first considered. The spin σ_{ik} is defined as the addition to the orbital angular momentum m_{ik} which will result in a constant of motion. Proportionality of momentum p_k and velocity v_k gives $\dot{m}_{ik}=0$ for a free particle. Thus the spin variables are defined by $u_k=\mu v_k-p_k$, μ a constant. A wave-equation is derived in the usual manner. Spin-space is assumed continuous with four variables ξ^k analogous to x^k with the same Lorentz transformations. Operators $\eta_k=(\hbar/i)\partial/\partial\xi^k$, $\omega_{ik}=\xi_i\eta_k-\xi_k\eta_i$ are introduced with $\sigma_{ik}=\hbar\omega_{ik}$. The square of the total spin has eigenvalues $l(l+1)\hbar^2$, $l=0, 1, 2, \cdots$. There is a "rest momentum" in addition to a rest mass and collisions may change the spin of a particle. Equations agreeing with those of Fierz [Helvetica Phys. Acta **12**, 3–37 (1939)] are obtained but with rest mass depending on spin. The theory is extended to half-integral spins by using spin representations in ξ-space.

C. Strachan (Aberdeen).

Petiau, Gérard. Sur un formalisme de la théorie des corpuscules de spin 0 ou $\hbar$ et leurs équations d'ondes dans les champs extérieurs. J. Phys. Radium (8) 12, 112–122 (1951).

From the first-order wave equations of de Broglie for a sixteen-component wave-function for a particle of maximum spin $\hbar$ and mass m_0 are derived second-order equations for an eight-component wave-function. Two of the latter describe a scalar or pseudoscalar meson, the other six a vector meson. Operators corresponding to velocity, angular momentum integrals, are discussed. Transformations under the Lorentz group are derived for fields with which the particle may be coupled. *C. Strachan* (Aberdeen).

Tani, Smio. Connection between particle models and field theories. I. The case spin 1/2. Progress Theoret. Physics 6, 267–285 (1951).

The application of the transformation found by Pryce [Proc. Roy. Soc. London. Ser. A. 195, 62–81 (1948); these Rev. 10, 662], Foldy and Wouthuysen [Physical Rev. (2) 78, 29–36 (1950)] and the author to the interaction of fermions with other fields is considered. Very simple semi-classical interpretations of the various effects are readily found using the new representation. Unfortunately these intuitive pictures are obtained at the cost of increasing the complexity of calculations. The transforms of the important spinor operators are tabulated. *K. M. Case.*

Tomonaga, Sin-itiro. Remarks on Bloch's method of sound waves applied to many-Fermion problems. Progress Theoret. Physics 5, 544–569 (1950).

This paper deals with an idea implied in the work of F. Bloch [Z. Physik 81, 363–376 (1933); Helvetica Phys. Acta 7, 385–405 (1934)], that it is possible to describe an assembly of Fermi particles by means of sound waves in the Fermi gas and hence by an equivalent assembly of Bose particles, the sound quanta. Treating the one-dimensional case of an assembly of noninteracting Fermi particles, the author shows that, if certain conditions are fulfilled, the Fourier components of the density satisfy simple commutation relations and linear equations of motion. Momenta conjugate to the density components are introduced and a suitable (quadratic) Hamiltonian is found, so that one has a formalism describing an assembly of Bose particles. Interactions between pairs of Fermi particles can be taken into account by the addition to the Hamiltonian of quadratic terms, so that the field equations remain linear. It is shown that this Hamiltonian is essentially equal to the original Hamiltonian of the Fermi particles. The energy eigen-value problem is solved by a principal-axis transformation of the Hamiltonian. The author discusses the correlation between particles, the effect of an external perturbing force, and the criteria for the applicability of the method. The method is valid if the states of the excited particles and of the holes in the Fermi gas lie within a certain interval and if the forces between particles are not of too short a range: roughly speaking, the range should be greater than four times the mean distance between particles. *N. Rosen.*

***Mariani, J. Metrical Geometry, Contact Transformations and Nuclear Physics.** Institute of Applied Biology and Cancer Research and Hospital Foundation, Brooklyn, N. Y., 1950. 28 pp.

The author points out that a two-dimensional unimodular transformation, and in particular a two-dimensional proper rotation, is a contact transformation. Therefore, a one-parameter group of such transformations can be described by a Hamiltonian. Quantization by the usual procedure leads to the introduction of a fundamental length below which our ordinary notions of geometry become meaningless. A classically fixed point, here becomes a Gaussian probability cloud. Applying his theory to elementary particles, the author is led to Born's equation for their rest masses [Born, Rev. Modern Physics 21, 461–473 (1949)]. *A. J. Coleman* (Toronto, Ont.).

Jost, R., and Pais, A. On the scattering of a particle by a static potential. Physical Rev. (2) 82, 840–851 (1951).

The radius of convergence of the perturbation method "the Born approximation" for the solution of scattering problems involving static potentials was studied. It is shown that it is indeed limited and for some potentials there is no energy range in which the perturbation method is useful. A completely convergent result is obtained by employing the Fredholm solution for the integral equation describing the scattering. It appears, however, that the Fredholm method is not too useful because of the slowness with which it converges. *H. Feshbach* (Cambridge, Mass.).

Ekstein, H. Multiple elastic scattering and radiation damping. I. Physical Rev. (2) 83, 721–729 (1951).

The integral equation for potential scattering in momentum space is transformed in such a way that the influence of radiation damping is separated. An analogous procedure is proposed for multiple scattering. Further generalization is achieved by eliminating the potential and replacing it by a function in momentum space; this function is still characteristic for the scattering properties of the individual scatterer, but it can also be used for an approximate treatment of cases where the single scattering process is no longer describable by a potential, e.g., absorption and inelastic scattering. A method of solving the integral equation for multiple scattering by iteration is outlined and compared with Born's approximation. As an application, scattering of thermal neutrons by a small crystal is considered. In this case, the second approximation is entirely due to radiation damping, and does not lead to a second order correction to the scattered intensity. It is necessary to calculate the third approximation in order to obtain the first non-vanishing correction. The range of validity of the first approximation is larger than according to previous estimates. *E. Gora* (Providence, R. I.).

Fogel, Karl-Gustav. On the pseudoscalar symmetrical meson theory of nuclear forces. Soc. Sci. Fenn. Comment. Phys.-Math. 14, no. 13, 32 pp. (1949).

This thesis shows that for meson masses less than 355 m_e the pseudoscalar symmetrical meson theory of nuclear forces cannot simultaneously explain the observed low energy $n-p$ scattering, the binding energy and quadripole moment of the deuteron. Further, in the Born approximation, this theory predicts a strong backward scattering for high energy $n-p$ scattering, in seeming contradiction with experiment as is also the predicted total cross-section. Finally, the mixed (pseudoscalar+vector) theories of Møller and Rosenfeld and of Møller, Rosenfeld and Schwinger are applied to calculate high energy $n-p$ scattering in Born approximation and also give disagreement with the observed total cross-section. *A. J. Coleman* (Toronto, Ont.).

Flint, H. T., and Williamson, E. Marjorie. The quantum mechanics of the electron. Proc. Roy. Soc. London. Ser. A. 207, 380–388 (1951).

The authors start from an analogy with the quantum-mechanical form of the law of conservation of angular momentum of an electron. In order to preserve the law it is found necessary to add the spin term to those which represent the orbital angular momentum. It can be shown that by considering the angular momentum tensor as a five-dimensional quantity the familiar expression of the angular momentum appears as part of a more general tensor which unites angular momentum and coordinate operators. The authors regard any five-dimensional vector as a union of a four-dimensional vector and a scalar quantity. The fifth covariant component A_5 of a contravariant vector A^μ ($\mu=1, 2, 3, 4, 5$) therefore can be regarded as a scalar in the four-dimensional continuum. In the case of the coordinate vector x^μ, the components x^1, x^2, x^3, x^4 form a four-dimensional vector, while x_5 is a scalar invariant length. Thus the ordinary coordinates used to describe the location of a particle must be supplemented by the addition of a fifth component, of which the covariant form is an invariant length characteristic of the particle. The fundamental lengths of particles, therefore, are of the nature of additional coordinates. If the indices m, n, l of the angular momentum operator M^{mnl} take the values 1, 2, 3, 4, 5 the components M^{k5} ($k=1, 2, 3$) can be written $M^{k5}=-im_0X^k$. Neglecting the effect of gravitation the authors get

$$X_k=x_k-\frac{x_5}{m_0c}u_k-\frac{hi}{4\pi m_0}\beta_k\beta_5$$

(β Dirac's matrices, m_0 rest mass, u_k momentum). The authors suggest that the operators X^k are the quantum form of coordinates. In the case of the angular momentum the quantum theory suggests that the classical concept requires supplementing by a quantity representing a characteristic property of the particle. Similarly it may be necessary in the use of coordinates to describe the position of a particle to include a term to represent an intrinsic length, given by the third term in the expression for X_k. The coordinates (X_k) and the momentum (u_k) are retained with their usual meaning and regarded as parts of an operator of the mechanical theory of the electron. But it is suggested that they are insufficient as a basis of mechanical description of the electron. The mechanical quantities, therefore, must be obtained by operations upon (X_k) and not upon (x_k). The momentum (u_k) like (x_k) also is part of an operator (M_k) of wider significance: $M_m=i\beta_4-m_0\dot{X}_m$. Finally the authors proceed to a force operator on an electron situated in an electromagnetic field by evaluation of the derivative of the operator M_m with respect to time. The first term of the resulting expression is the Lorentz force operator ($k=1, 2, 3$). The additional second term arises proportional to the derivative of the acceleration with respect to time. This term must be considered as a non-Maxwellian quantity interacting with the external field. *M. Pinl* (Dacca).

Jackson, J. L. A variational approach to nuclear reactions. Physical Rev. (2) 83, 301–304 (1951).

The nuclear reaction theory of Wigner and Eisenbud [Physical Rev. (2) 72, 29–41 (1947)] introduced a certain real symmetric matrix R with rows and columns labelled by the possible alternatives of the reaction. In the present paper an expression for the R matrix is obtained on the basis of a variational calculation. The variation principle is of the same general type as that developed by the reviewer [Kungl. Fysiografiska Sällskapets i Lund Förhandlingar 14, no. 21 (1944); cf. also C. R. Dixième Congrès des Mathématiciens Scandinaves, Copenhagen, 1946, pp. 201–206; these Rev. 6, 111; 8, 425] for the two-body quantum-mechanical scattering problem. The actual derivation follows lines resembling those of W. Kohn [Physical Rev. (2) 74, 1763–1772 (1948)]. The variation principle is applied to a trial function, formed by a linear combination of a finite number of the orthonormal resonance functions X_λ, introduced by Wigner and Eisenbud. By varying the coefficients of the X_λ's one obtains directly an expression for the R matrix in terms of the states represented by the functions X_λ, which agrees with the rigorous expression for the R matrix up to the order of the trial function used. Consequently the procedure converges to the true solution.

L. Hulthén (Stockholm).

Jauho, Pekka. On the unique determination of the nuclear potential between charged nucleons with the aid of scattering experiments. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 80, 44 pp. (1950).

The nuclear potential is shown to be uniquely determined by the nuclear phase if that phase is such that no bound states can occur and if the potential is assumed to go to zero rapidly enough at infinity; $r^2V(r)$ must also approach zero with r. Non-relativistic equations, Coulomb and central nuclear potentials, and chiefly S states in proton-proton scattering are considered. The method is based on Levinson's idea of mixed Green's functions. *T. E. Hull.*

Vrkljan, Vladimir. Befriedigen die de Broglie-schen Wellengleichungen das Relativitätsprinzip? Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 5, 198–201 (1950). (Serbo-Croatian. German summary)

Cave, L., Corner, J., and Liston, R. H. A. The scattering of gamma-rays in extended media. I. Perpendicular incidence on a plane slab. Proc. Roy. Soc. London. Ser. A. 204, 223–259 (1950).

Gamma rays of a certain frequency ν are normally incident on a plane slab of uniform thickness, the number of incident quanta per unit area of the slab being the same at all points of the face. The gamma rays suffer Compton scatterings in the slab in accordance with the Klein-Nishina formula (for unpolarized radiation) and are degraded in energy. It is required to find the density of the radiation, $\int I(x, \mu, \phi; E)d\omega$, with a photon energy E ($\theta=\cos^{-1}\mu$ and ϕ are the polar angles referred to the outward normal and $d\omega$ is the element of solid angle) at various depths, x, and also the dose defined by $mc^3\int dE\sigma_a(E)\int I(x, \mu, \phi; E)d\omega$ where $\sigma_a(E)$ is the cross-section for energy absorption by Compton scattering which is a function of the energy E expressed in units of mc^3. Expressions are given by $I(x, \mu, \varphi; E)$ resulting from once scattered and twice scattered radiations and the corresponding values for the doses (expressed as $\epsilon^{-\alpha D}$ where D is the thickness of the slab in units of the mean free path of the incident quantum and α is a function of D and $\gamma=h\nu/mc^3$) are graphically illustrated. An upper limit for the dose was also derived by a method which the authors describe in the following terms. "We start with the relative dose $\epsilon^{-\alpha D}$ at thickness D mean free paths, where α depends on γ and D. Of this dose ϵ^{-D} comes from radiation which has not been scattered and $\epsilon^{-\alpha D}-\epsilon^{-D}$ from radiation scattered at least once. We now write down the scattered radiation arriving at a point on the far side of the slab, labelling the

quanta according to the point at which they last had a collision. At such a point we know the unscattered component in direction, intensity and wave-length, and we know the scattered radiation only as regards total dose. We assume an average direction and wave-length for this scattered component and proceed to calculate the total scattered radiation at a point on the far side of the slab. It remains to choose the average direction and wave-length, already mentioned, to maximize the dose at the far side of the slab. The simplest method is to assume the original direction and initial $h\nu/mc^2$. It can be shown that this fails to maximize the dose for small D but ought to be suitable at large D. Finally, the scattered dose $e^{-aD}-e^{-D}$ is obtained as an integral involving the function α for the initial value of $h\nu/mc^2$ and all thicknesses up to D. This integral equation has been converted to an integro-differential equation and solved numerically for $\gamma=10$." An approximate solution for the problem was also obtained by the following method: The equation of transfer for the problem is written down, neglecting the degradation of energy in scattering and assuming that scattering takes place in accordance with a phase function proportional to $e^{-\beta(1-\mu)}$ where β is a constant. "The unscattered radiation is split off and an equation of transfer is obtained for the intensity of radiation which has been scattered at least once. A plausible form with two arbitrary functions is assumed for this intensity and inserted in the transfer equation, which is expanded in powers of $(1-\mu)$. The terms in $(1-\mu)^0$ and $1-\mu$ give two differential equations which can be solved for the arbitrary functions. Finally, the relative dose is obtained by an integral over the scattered intensity and the usual term from the unscattered radiation. The success of the method depends on guessing a good form for the scattered intensity. The form we used was $J(x,\mu)=(\beta/2\pi)A(x)\exp\{-N\sigma x-\beta(1-\mu)B(x)\}$ for the scattered intensity J at deflexion μ $(=\cos\theta)$ and depth x into a semi-infinite block; here $A(x)$ and $B(x)$ are the two arbitrary functions. This solution is not applicable at small values of $N\sigma x$, since close to the surface any scattered radiation is bound to be directed outwards. At bigger depths this difficulty disappears, and the form chosen has the correct tendency to favour radiation going on with small deflexions." The calculations for the dose based on this approximation are also illustrated.

S. Chandrasekhar (Williams Bay, Wis.).

Kushneriuk, S. A., and Preston, M. A. The scattering phase shifts of two-nucleon systems. Proc. Phys. Soc. Sect. A. 64, 712–725 (1951).

Certain simple functions of the scattering phase shifts for nucleons are expanded in powers of the energy. Methods due to Bethe are extended to the cases of general orbital angular momenta for central forces in neutron-proton and proton-proton scattering and for non-central static forces in proton-proton scattering including coupling of P and F states. The differential scattering cross section for this latter case is also given. *T. E. Hull* (Vancouver, B. C.).

Thirring, W., and Touschek, B. A covariant formulation of the Bloch-Nordsieck method. Philos. Mag. (7) 42, 244–249 (1951).

The Bloch-Nordsieck results on the multiple production of low energy quanta in the scattering of particles are derived in a covariant manner from the field equations of Yang and Feldman [Physical Rev. (2) 79, 972–978 (1950); these Rev. 12, 569] and also from the Dyson-Feynman form of the S matrix. *C. Strachan* (Aberdeen).

Thermodynamics, Statistical Mechanics

***Hinčin, A. Ya. On the analytical apparatus of physical statistics.** Trudy Mat. Inst. Steklov. 33, ii+54 pp. (1950). (Russian)

The author has shown how one can base the asymptotic formulas of classical statistical mechanics on the central limit theorem of probability theory [Mathematical Foundations of Statistical Mechanics, Moscow-Leningrad, 1943; these Rev. 8, 187; or, in translation, Dover, New York, 1949; these Rev. 10, 666]. In the present paper he carries through the corresponding development for quantum statistics, already sketched in the Maxwell-Boltzmann case in an earlier paper [Izvestiya Akad. Nauk SSSR. Ser. Mat. 7, 167–184 (1943); these Rev. 5, 280]. It is supposed that the given physical system consists of N components of identical structure, with weak interactions, in a volume of magnitude V. The system energy is E, and it is supposed that V, N, E are large, of the same order. Let $a_r(U)$ be the number of components in a specified stationary state U for which the component energy is ϵ_r, and let $\Omega(N,E)$, defining the "structure function" of the system, be the number of linearly independent states of the system for given N, E, V. The first task of the theory is to evaluate the microcanonical expected values of additive quantities, that is, those which can be expressed as sums of corresponding values for each component. In particular, one must evaluate $\bar{a}_r=\sum_U a_r(U)/\Omega(N,E)$, by means of which the general case can be treated. The sum is extended over the states U corresponding to a complete orthonormal system of admissible state functions. The averages $\overline{a_r}$, $\overline{a_r a_s}$ are computed for Maxwell-Boltzmann, Bose-Einstein, and Fermi-Dirac statistics, in terms of the structure function. The problem is then to obtain asymptotic expressions for the structure function. It is supposed that V is an integer, and that the component energy levels are integers, with Vg_k linearly independent states at energy level k. For each k let x_k', y_k' be integer-valued random variables with $y_k'=kx_k'$, and $\Pr\{x_k'=n\}=c\gamma(n)e^{-(\alpha+\beta k)n}$, where α, β are positive constants, specified below, c is a proportionality constant, and $\gamma(n)$ is (respectively for the three statistical systems listed above): $1/n!$; 1; 1 for $n\leqq 1$, 0 otherwise. Now let $\{(x_n,y_n),\ n\geqq 1\}$ be a sequence of mutually independent pairs of random variables, containing, for each k, Vg_k pairs with the distribution of (x_k',y_k'), and let $S_V=\sum x_n$, $T_V=\sum y_n$. Then it is shown that

$$\Omega(p,q)=\Phi(\alpha,\beta)e^{\alpha p+\beta q}\Pr\{S_V=p,\ T_V=q\},$$

where

$$\Phi(\alpha,\beta)=\sum_{x,y=0}^{\infty}e^{-\alpha x-\beta y}\Omega(x,y).$$

Now (S_V,T_V) is the sum of V pairs of random variables with the distribution of (S_1,V_1). Thus the evaluation of the structure function is reduced to the evaluation of the distribution of the sum of a large number of two-dimensional random variables whose values are integer pairs. The parameters α, β are determined in such a way that S_V, T_V have expectations N, E respectively, and the local central limit theorem is then used to evaluate the distribution of (S_V,T_V) for large V. The desired expectations are then obtained. The expected value and dispersion of any additive quantity both have the form const. $\times V+O(V^{1/2})$, so that such a quantity is relatively near its expected value, with large probability.

The necessary modifications in this theory are made to treat photons, in which case N is itself a random variable.

In the Maxwell-Boltzmann case the theory can be modified in such a way that, just as in the classical case, only the one-dimensional central limit theorem, rather than the two-dimensional one, is needed. Systems with components of two types are treated in detail. In particular, if the number of components of type 1 is small compared with the number of type 2, the system of components of type 1 is subject to the canonical Gibbsian statistics, in which the energy is a random variable. It is shown that the error in using canonical rather than microcanonical averages is of order $V^{-1/2}$. [See also the paper reviewed below, in which the author finds the actual distributions of the a_r's.] *J. L. Doob.*

Hinčin, A. Ya. On the distribution laws of the "occupation numbers" in quantum statistics. Doklady Akad. Nauk SSSR (N.S.) **78**, 461–463 (1951). (Russian)

The author finds the distributions of the occupation numbers, the $a_r(U)$ of the preceding review. More exactly, he finds the limiting microcanonical distributions, as the number of components of the system under examination becomes infinite. In the Maxwell-Boltzmann case the limiting distribution of $a_r(U)$ is Poisson; in the Bose-Einstein case it is geometric. In the Fermi-Dirac case the problem is trivial, since $a_r(U)$ can then take on only the values 0 and 1, so that its distribution is determined by its expectation, which is known. *J. L. Doob* (Urbana, Ill.).

***Placzek, G. Correlation of position for the ideal quantum gas.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 581–588. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

Wakefield, A. J. Statistics of the simple cubic lattice. II. Proc. Cambridge Philos. Soc. **47**, 799–810 (1951).

In a previous paper [same Proc. **47**, 419–435 (1951); these Rev. **13**, 308] the author has given series for two functions $Z(z)$ and $U(u)$ such that, for low temperatures, the partition function can be expressed in terms of $Z(z)$ for $z=\exp(-2J/kT)$ and, for high temperature, the partition function can be expressed in terms of $U(u)$ for $u=(1-z)/(1+z)$. The function $Z(z)$ is an even function of z and the author gives the series expansion up to the term involving z^{28}. The function $U(u)$ is also even and its expansion is given up to the term involving u^{12}. The author studies the coefficient of the U series and shows that the coefficient of z^{2n} can be expressed very accurately in the form $C\gamma^n/(2n-1)2n(2n+1)$ and consequently $U(u)$ can be given by the formula,

$$U(u)=.497[(au+(au)^{-1})\log((1+au)/(1-au)) +2\log(1-a^2u^2)-2]+S(u)$$

where $a^2=20.9$. This yields a critical value corresponding to $z=.641$. The treatment of $Z(z)$ is somewhat more complicated. The coefficients of z^{2n} are expressed as a sum of two terms α_n and β_n such that α_n is large asymptotically compared with β_n but the latter is not negligible. The result is

$$Z(z)=0.1937\{(bz+(bz)^{-1})\tan^{-1}bz-\tfrac{1}{2}\} +7.5\left\{\tfrac{1}{2}(cz+(cz)^{-1})\log\frac{1+cz}{1-cz}+\tfrac{1}{2}\log(1-c^2z^2)-\tfrac{1}{2}\right\}+R'(z),$$

where $1/b=.5326$ and $1/c=.641$ has been selected to give consistency with the U result. The expression involving b results from the α_n terms and is analytic at $z=1/b$ although this is the radius of convergence of the original z series. The resulting expressions for the partition function are used to study the energy, specific heats and spontaneous magnetization by the customary thermodynamical procedures. *F. J. Murray* (New York, N. Y.).

Syôzi, Itiro. Statistics of kagomé lattice. Progress Theoret. Physics **6**, 306–308 (1951).

The "kagomé" lattice is a plane pattern whose statistics are obtained by two transformations which yield a honeycomb lattice. The latter has been solved by the basic Onsager method. *F. J. Murray* (New York, N. Y.).

Frank, F. C. Crystal dislocations. Elementary concepts and definitions. Philos. Mag. (7) **42**, 809–819 (1951).

The purpose of this paper is to define the concepts of crystal dislocation theory in general form, i.e., independent of special crystal models or lattices. Linear dislocations are stressed and the associated Burgers circuits which correspond to closed circuits in the undeformed lattice. When the linear dislocation is surrounded by "good" crystal, it is said to be perfect. Imperfect linear dislocations are associated with certain "twinning" surfaces. The motion and growth of these dislocations are discussed in particular in relation to the free energy and possible applied stresses. The thermodynamical considerations based on the free energy introduce statistical concepts associated with the topological and geometrical notions of the crystal lattice and focus interest on certain possibilities in the formulation of the theory. References are given to many recent papers on crystal theory to indicate the basic unifying principles. *F. J. Murray* (New York, N. Y.).

Kikuchi, Ryoichi. A theory of cooperative phenomena. II. Equation of states for classical statistics. J. Chem. Phys. **19**, 1230–1241 (1951).

The author proposes a method of finding the equation of states of classical statistics based on the statistical methods for studying lattices described in his previous papers [cf. Physical Review. (2) **81**, 988–1003 (1951); these Rev. **12**, 788]. These methods involve building up the lattice in a step-by-step fashion. This process can be applied to the situation in which one has a number of intertwined lattices. "The solid state is one in which atoms exist chiefly in one of the . . . sublattices. In the fluid phases the probability of existence of an atom on each of the . . . sublattices is equal". The author solves the one-dimensional problem by this method taking the limit as the number of intertwined lattices approach infinity and gives formulas for treating the higher dimensional cases. For the solid state the equivalent of the Bethe approximations is given. *F. J. Murray.*

Frisch, H. L., Collins, F. C., and Friedman, B. Excluded volume effect in polymer chains. I. J. Chem. Phys. **19**, 1402–1409 (1951).

Polymer chains correspond to non-overlapping random walks. These walks are considered for a square lattice and hexagonal lattice in the plane and a diamond lattice in space. A vectorial description of a random walk is given and a certain subclass of the class of non-overlapping random walks on each of the above lattices is specified. The number of elements in this subclass for a given length n is then determined by means of a difference equation. This yields

an underestimate of the total number of non-overlapping paths with n steps. On the other hand, a linear system of difference equations can be set up for the number of paths on a square lattice in which no overlap occurs in the last a steps taken. The solution of this system depends, for a given a, on a characteristic value problem. *F. J. Murray.*

BIBLIOGRAPHICAL NOTES

***Herland, Leo. Dictionary of Mathematical Sciences. Volume I. German-English. Wörterbuch der mathematischen Wissenschaften. Band I. Deutsch-Englisch.** Frederick Ungar Publishing Co., New York, N. Y., 1951. ii+235 pp. $3.25.

Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U. S. A., 1950.

These Proceedings, published by the American Mathematical Society, appeared in two volumes in March, 1952. Vol. I [v+769 pp.] contains the list of officers, delegates and members, the report of the secretary, the stated addresses, and the addresses and communications in the various sections. Vol. II [iii+461 pp.] contains the reports of the conferences in algebra, analysis, applied mathematics, and topology. Abstracts and reports stating that a detailed version will appear later will not be reviewed separately.

Atti del Terzo Congresso dell'Unione Matematica Italiana.

The proceedings of this congress, held in Pisa, 23–26 September 1948, have been published by the Case Editrice Perrella [Rome, 1951]. Most of the papers, which are in the nature of short abstracts, will not be reviewed separately.

Premier Congrès des Mathématiciens et Physiciens de la R. P. F. Y.

This congress was held 8–12 November 1949 in Bled, Yugoslavia. The proceedings are published in two volumes [Naučna Knjiga, Belgrade, vol. 1, 1950, vol. 2, 1951]: vol. 1, Rapports et discussions; vol. 2, Communications et exposés scientifiques. Some of the papers from vol. 2 will be reviewed separately.

Proceedings of The First Pakistan Statistical Conference.

This conference was held in February, 1950 at the University of the Panjab, Lahore. The Proceedings, published by the university, appeared in 1951. Papers of mathematical interest will be reviewed separately. The Abstracts of Papers, which appears as a separate booklet, will not be reviewed separately.

All-Soviet Topological Conference (1950).

Abstracts of the lectures given at this conference (held 26–30 May 1950) are contained in Uspehi Matem. Nauk (N.S.) 6, no. 4(44), 193–220 (1951). They will not be reviewed separately.

Elenco delle pubblicazioni dell'Istituto Nazionale per le Applicazioni del Calcolo. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 305, 24 pp. (1951).

The publications in this list are divided into two series. Those in Serie I span the period 1927–1936 and are arranged alphabetically by author. Those in Serie II are listed according to a number assigned by the Institute which is approximately chronological and span the period 1936–1950 (nos. 1–302).

Elenco dei lavori eseguiti o in via di esecuzione nell'Istituto Nazionale per le Applicazioni del Calcolo dal 1937 all'Aprile 1951. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. 304, 40 pp. (1951).

A list by years of the various projects of the Institute of the title, together with the name of the organization or individual proposing the work.

Analele Academiei Republicii Populare Române. Secțiunea de Științe Matematice, Fizice și Chimice. Seria A.

Volume 1 of these annals began in 1948.

Journal d'Analyse Mathématique.

Vol. 1 is dated 1951. The address of the editorial office is 24, Rue Ibn Ezra, Jerusalem, Israel.

Journal of Rational Mechanics and Analysis.

Vol. 1, no. 1, of this journal appeared in January, 1952. It is published by The Graduate Institute for Applied Mathematics, Indiana University, Bloomington, Indiana.

Memoirs of the College of Science, University of Kyoto.

Starting with vol. 26, Series A has split into two sections, Series A, Mathematics, and Series A (with no further designation).

Pacific Journal of Mathematics.

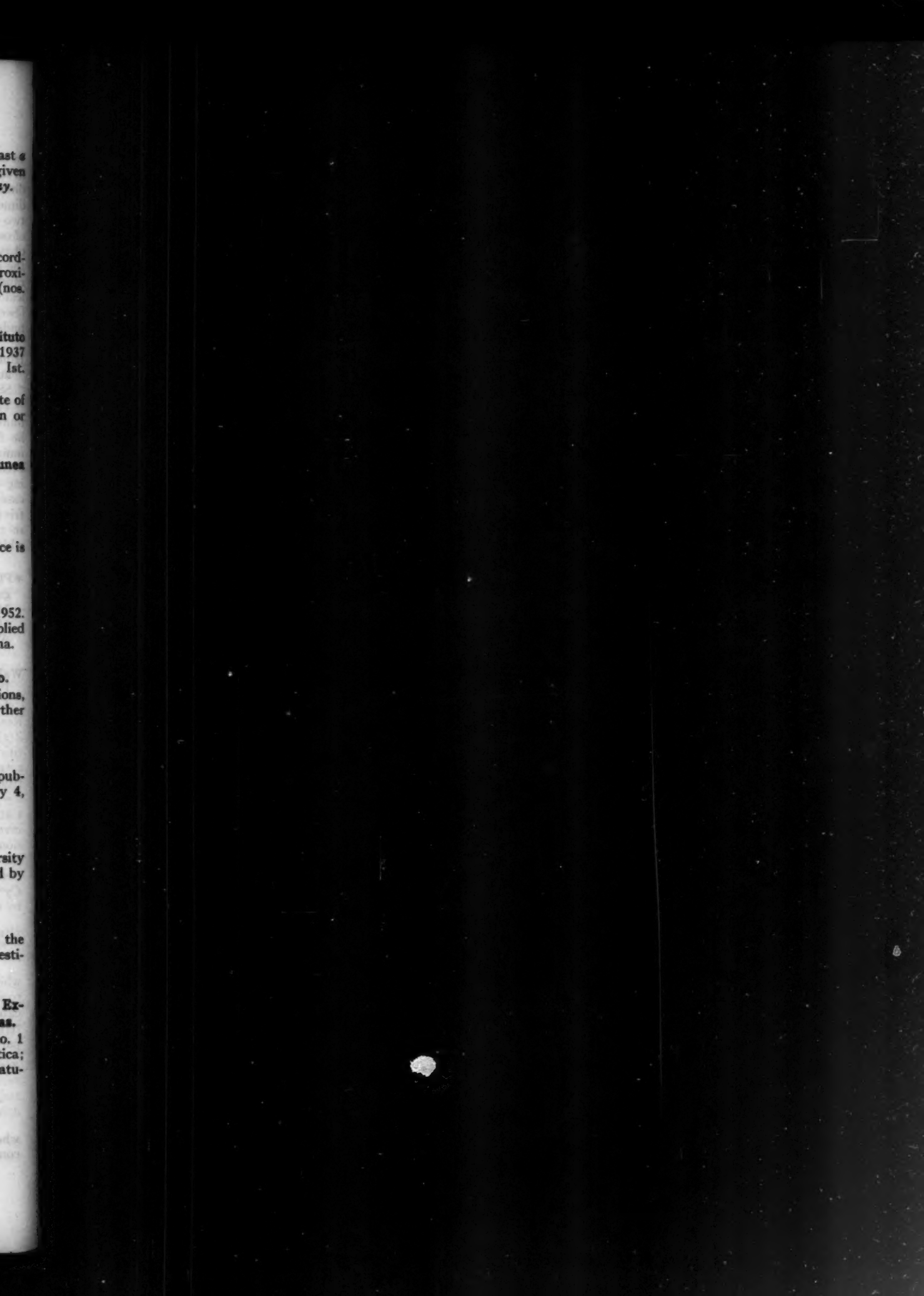
Vol. 1, no. 1, is dated March, 1951. The journal is published by the University of California Press, Berkeley 4, Calif.

Proceedings of the London Mathematical Society.

Vol. 1 of a third series, published by the Oxford University Press, was issued in 1951. The second series, published by C. F. Hodgson and Son, will terminate with vol. 54.

Trabajos de Estadística.

Vol. 1, no. 1, appeared in 1950. It is published by the Departamento de Estadística, Consejo Superior de Investigaciones Científicas, Madrid.

Universidad de Buenos Aires. Facultad de Ciencias Exactas, Físicas y Naturales. Contribuciones Científicas.

These Contribuciones Científicas, of which vol. 1, no. 1 is dated 1950, appear in five series: Serie A: Matemática; Serie B: Física: Serie C: Química; Serie D: Ciencias Naturales; Serie E: Geología.